Three-Dimensional Kinematics

Position

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P \qquad \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}^P = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \begin{bmatrix} a_{i11} & a_{i12} & a_{i13} \\ a_{i21} & a_{i22} & a_{i23} \\ a_{i31} & a_{i32} & a_{i33} \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}'^P$$

Velocity

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [\dot{A}_i]\{s_i\}'^P \qquad \{\dot{r}_i\}^P = \{\dot{r}_i\} + \{\omega_i\} \times \{s_i\}^P$$

$$\{\omega_i\} = \begin{Bmatrix} \omega_{ix} \\ \omega_{iy} \\ \omega_{iz} \end{Bmatrix} \qquad [\tilde{\omega}_i] = \begin{bmatrix} 0 & -\omega_{iz} & \omega_{iy} \\ \omega_{iz} & 0 & -\omega_{ix} \\ -\omega_{iy} & \omega_{ix} & 0 \end{bmatrix} \qquad \{\omega_i\} \times \{s_i\}^P = [\tilde{\omega}_i]\{s_i\}^P$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [\tilde{\omega}_i]\{s_i\}^P = \{\dot{r}_i\} + [\tilde{\omega}_i][A_i]\{s_i\}'^P = \{\dot{r}_i\} + [A_i][\tilde{\omega}_i]'\{s_i\}'^P$$

$$[\dot{A}_i] = [\tilde{\omega}_i][A_i] = [A_i][\tilde{\omega}_i]'$$

$$\{s_i\}^P = [A_i]\{s_i\}'^P \qquad \{s_i\}'^P = [A_i]^T\{s_i\}^P$$

$$\{\omega_i\} = [A_i]\{\omega_i\}' \qquad \{\omega_i\}' = [A_i]^T\{\omega_i\}$$

$$[\tilde{\omega}_i] = [A_i][\tilde{\omega}_i]'[A_i]^T \qquad [\tilde{\omega}_i]' = [A_i]^T[\tilde{\omega}_i][A_i]$$

Acceleration

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [\ddot{A}_i]\{s_i\}'^P$$

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [\dot{\tilde{\omega}}_i][A_i]\{s_i\}'^P + [\tilde{\omega}_i][\dot{A}_i]\{s_i\}'^P = \{\ddot{r}_i\} + [\dot{\tilde{\omega}}_i][A_i]\{s_i\}'^P + [\tilde{\omega}_i][\tilde{\omega}_i][A_i]\{s_i\}'^P$$

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [\beta_i][A_i]\{s_i\}'^P \qquad [\beta_i] = [\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i]$$

$$[\ddot{A}_i] = \left([\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i]\right)[A_i] = [\beta_i][A_i]$$

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [\dot{A}_i][\tilde{\omega}_i]'\{s_i\}'^P + [A_i][\dot{\tilde{\omega}}_i]'\{s_i\}'^P = \{\ddot{r}_i\} + [A_i][\tilde{\omega}_i]'[\tilde{\omega}_i]'\{s_i\}'^P + [A_i][\dot{\tilde{\omega}}_i]'\{s_i\}'^P$$

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [A_i][\beta_i]'\{s_i\}'^P \qquad [\beta_i]' = [\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'$$

$$[\ddot{A}_i] = [A_i]\left([\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'\right) = [A_i][\beta_i]'$$

$$\{\dot{\omega}_i\} = [A_i]\{\dot{\omega}_i\}' \qquad \{\dot{\omega}_i\}' = [A_i]^T\{\dot{\omega}_i\}$$

$$[\dot{\tilde{\omega}}_i] = [A_i][\dot{\tilde{\omega}}_i]'[A_i]^T \qquad [\dot{\tilde{\omega}}_i]' = [A_i]^T[\dot{\tilde{\omega}}_i][A_i]$$

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$$\{\dddot{r}_i\}^P = \{\dddot{r}_i\} + [\dddot{A}_i]\{s_i\}'^P$$

$$\{\dddot{r}_i\}^P = \{\dddot{r}_i\} + \left([\ddot{\tilde{\omega}}_i] + 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i]\right)[A_i]\{s_i\}'^P$$

$$[H_i] = 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i]$$ NOT angular momentum

$$[H_i] = [A_i]\left([H_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' - [\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]'\right)[A_i]^T$$

$$\{\dddot{r}_i\}^P = \{\dddot{r}_i\} + \left([\ddot{\tilde{\omega}}_i] + [H_i]\right)[A_i]\{s_i\}'^P$$

$$[\dddot{A}_i] = \left([\ddot{\tilde{\omega}}_i] + 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i]\right)[A_i]$$

$$\{\dddot{r}_i\}^P = \{\dddot{r}_i\} + [A_i]\left([\ddot{\tilde{\omega}}_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'\right)\{s_i\}'^P$$

$$[H_i]' = 2[\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'$$ NOT angular momentum

$$[H_i]' = [A_i]^T\left([H_i] - [\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i]\right)[A_i]$$

$$\left\{\dddot{r}_i\right\}^P = \left\{\dddot{r}_i\right\} + [A_i]\left(\left[\ddot{\tilde{\omega}}_i\right]' + [H_i]'\right)\left\{s_i\right\}'^P$$

$$\left[\dddot{A}_i\right] = [A_i]\left(\left[\ddot{\tilde{\omega}}_i\right]' + \left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'\right)$$

$$\left\{\ddot{\omega}_i\right\} = [A_i]\left([\tilde{\omega}_i]'\left\{\dot{\omega}_i\right\}' + \left\{\ddot{\omega}_i\right\}'\right) \qquad \left\{\ddot{\omega}_i\right\}' = [A_i]^T\left(\left\{\ddot{\omega}_i\right\} - [\tilde{\omega}_i]\left\{\dot{\omega}_i\right\}\right)$$

$$\left[\ddot{\tilde{\omega}}_i\right] = [A_i]\left(\left[\ddot{\tilde{\omega}}_i\right]' - \left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + [\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]'\right)[A_i]^T \qquad \left[\ddot{\tilde{\omega}}_i\right]' = [A_i]^T\left(\left[\ddot{\tilde{\omega}}_i\right] + \left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i] - [\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right]\right)[A_i]$$

Snap

$$\left\{\ddddot{r}_i\right\}^P = \left\{\ddddot{r}_i\right\} + \left[\ddddot{A}_i\right]\left\{s_i\right\}'^P$$

$$\left\{\ddddot{r}_i\right\}^P = \left\{\ddddot{r}_i\right\} + \begin{pmatrix} \left[\dddot{\tilde{\omega}}_i\right] + 3\left[\ddot{\tilde{\omega}}_i\right][\tilde{\omega}_i] + 3\left[\dot{\tilde{\omega}}_i\right]\left[\dot{\tilde{\omega}}_i\right] + [\tilde{\omega}_i]\left[\ddot{\tilde{\omega}}_i\right] \\ +3\left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i][\tilde{\omega}_i] + 2[\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i] + [\tilde{\omega}_i][\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right] \\ +[\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i] \end{pmatrix}[A_i]\left\{s_i\right\}'^P$$

$$[W_i] = 3\left[\ddot{\tilde{\omega}}_i\right][\tilde{\omega}_i] + 3\left[\dot{\tilde{\omega}}_i\right]\left[\dot{\tilde{\omega}}_i\right] + [\tilde{\omega}_i]\left[\ddot{\tilde{\omega}}_i\right] + 3\left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i][\tilde{\omega}_i] + 2[\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i] + [\tilde{\omega}_i][\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i]$$

$$[W_i] = [A_i]\left([W_i]' + 2\left[\ddot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' - 2[\tilde{\omega}_i]'\left[\ddot{\tilde{\omega}}_i\right]' + 2\left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' - 2[\tilde{\omega}_i]'[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]'\right)[A_i]^T$$

$$\left\{\ddddot{r}_i\right\}^P = \left\{\ddddot{r}_i\right\} + \left(\left[\dddot{\tilde{\omega}}_i\right] + [W_i]\right)[A_i]\left\{s_i\right\}'^P$$

$$\left\{\ddddot{r}_i\right\}^P = [A_i]\begin{pmatrix} \left[\dddot{\tilde{\omega}}_i\right]' + \left[\ddot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + 3\left[\dot{\tilde{\omega}}_i\right]'\left[\dot{\tilde{\omega}}_i\right]' + 3[\tilde{\omega}_i]'\left[\ddot{\tilde{\omega}}_i\right]' \\ +\left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + 3[\tilde{\omega}_i]'[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]' \\ +[\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' \end{pmatrix}\left\{s_i\right\}'^P$$

$$\begin{aligned}[W_i]' = {} & \left[\ddot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + 3\left[\dot{\tilde{\omega}}_i\right]'\left[\dot{\tilde{\omega}}_i\right]' + 3[\tilde{\omega}_i]'\left[\ddot{\tilde{\omega}}_i\right]' + \left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]'[\tilde{\omega}_i]' + 3[\tilde{\omega}_i]'[\tilde{\omega}_i]'\left[\dot{\tilde{\omega}}_i\right]' \\ & + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]'\end{aligned}$$

$$[W_i]' = [A_i]^T\left([W_i] - 2\left[\ddot{\tilde{\omega}}_i\right][\tilde{\omega}_i] + 2[\tilde{\omega}_i]\left[\ddot{\tilde{\omega}}_i\right] - 2\left[\dot{\tilde{\omega}}_i\right][\tilde{\omega}_i][\tilde{\omega}_i] + 2[\tilde{\omega}_i][\tilde{\omega}_i]\left[\dot{\tilde{\omega}}_i\right]\right)[A_i]$$

$$\left\{\ddddot{r}_i\right\}^P = \left\{\ddddot{r}_i\right\} + [A_i]\left([\tilde{\dddot{\omega}}_i]' + [W_i]'\right)\{s_i\}'^P$$

$$\{\dddot{\omega}_i\} = [A_i]\left(\{\dddot{\omega}_i\}' + 2[\tilde{\omega}_i]'\{\ddot{\omega}_i\}' + \left([\tilde{\dot{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'\right)\{\dot{\omega}_i\}'\right)$$

$$\{\dddot{\omega}_i\}' = [A_i]^T\left(\{\dddot{\omega}_i\} - 2[\tilde{\omega}_i]\{\ddot{\omega}_i\} - \left([\tilde{\dot{\omega}}_i] - [\tilde{\omega}_i][\tilde{\omega}_i]\right)\{\dot{\omega}_i\}\right)$$

$$[\tilde{\dddot{\omega}}_i] = [A_i]\begin{pmatrix} [\tilde{\dddot{\omega}}_i]' - 2[\tilde{\ddot{\omega}}_i]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'[\tilde{\ddot{\omega}}_i]' \\ + [\tilde{\dot{\omega}}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' - 2[\tilde{\omega}_i]'[\tilde{\dot{\omega}}_i]'[\tilde{\omega}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\dot{\omega}}_i]' \end{pmatrix}[A_i]^T$$

$$[\tilde{\dddot{\omega}}_i]' = [A_i]^T\begin{pmatrix} [\tilde{\dddot{\omega}}_i] + 2[\tilde{\ddot{\omega}}_i][\tilde{\omega}_i] - 2[\tilde{\omega}_i][\tilde{\ddot{\omega}}_i] \\ + [\tilde{\dot{\omega}}_i][\tilde{\omega}_i][\tilde{\omega}_i] - 2[\tilde{\omega}_i][\tilde{\dot{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\dot{\omega}}_i] \end{pmatrix}[A_i]$$

2D partial derivatives

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$$

by inspection $\left(\{r_i\}^P\right)_{ri} = [I_2]$ $\left(\{r_i\}^P\right)_{\phi i} = [B_i]\{s_i\}'^P$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + \dot{\phi}_i[B_i]\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = ([I_2])\{\dot{r}_i\} + \left([B_i]\{s_i\}'^P\right)\dot{\phi}_i + (0)t$$

chain rule $\{\dot{r}_i\}^P = \left(\{r_i\}^P\right)_{ri}\{\dot{r}_i\} + \left(\{r_i\}^P\right)_{\phi i}\dot{\phi}_i + \left(\{r_i\}^P\right)_t$

compare to terms in chain rule $\left(\{r_i\}^P\right)_{ri} = [I_2]$ $\left(\{r_i\}^P\right)_{\phi i} = [B_i]\{s_i\}'^P$ $\left(\{r_i\}^P\right)_t = 0$

3D partial derivatives

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [A_i][\tilde{\omega}_i]'\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [A_i]\left(\{\omega_i\}' \times \{s_i\}'^P\right) = \{\dot{r}_i\} - [A_i]\left(\{s_i\}'^P \times \{\omega_i\}'\right)$$

$\{\dot{r}_i\}^P = \{\dot{r}_i\} - [A_i][\tilde{s}_i]'^P\{\omega_i\}'$

$\{\omega_i\}' = 2[G_i]\{\dot{p}_i\}$

$\{\dot{r}_i\}^P = \{\dot{r}_i\} - 2[A_i][\tilde{s}_i]'^P[G_i]\{\dot{p}_i\}$

$\{\dot{r}_i\}^P = ([I_3])\{\dot{r}_i\} + (-2[A_i][\tilde{s}_i]'^P[G_i])\{\dot{p}_i\} + (0)t$

chain rule $\{\dot{r}_i\}^P = (\{r_i\}^P)_{ri}\{\dot{r}_i\} + (\{r_i\}^P)_{pi}\{\dot{p}_i\} + (\{r_i\}^P)_t$

compare to terms in chain rule $(\{r_i\}^P)_{ri} = [I_3]$ $(\{r_i\}^P)_{pi} = -2[A_i][\tilde{s}_i]'^P[G_i]$ $(\{r_i\}^P)_t = 0$

partial derivative wrt Euler parameters $(\{r_i\}^P)_{pi} = -2[A_i][\tilde{s}_i]'^P[G_i]$

$\{\dot{r}_i\}^P = \{\dot{r}_i\} - [A_i][\tilde{s}_i]'^P\{\omega_i\}'$

$\{\dot{r}_i\}^P = ([I_3])\{\dot{r}_i\} + (-[A_i][\tilde{s}_i]'^P)\{\omega_i\}' + (0)t$

chain rule $\{\dot{r}_i\}^P = (\{r_i\}^P)_{ri}\{\dot{r}_i\} + (\{r_i\}^P)_{\pi' i}\{\omega_i\}' + (\{r_i\}^P)_t$

compare to terms in chain rule $(\{r_i\}^P)_{ri} = [I_3]$ $(\{r_i\}^P)_{\pi' i} = -[A_i][\tilde{s}_i]'^P$ $(\{r_i\}^P)_t = 0$

partial derivative wrt π' directions $(\{r_i\}^P)_{\pi' i} = -[A_i][\tilde{s}_i]'^P$

$\lfloor *_{pi} \rfloor = 2\lfloor *_{\pi' i} \rfloor [G_i]$ $\lceil *_{\pi' i} \rceil = \frac{1}{2}\lceil *_{pi} \rceil [G_i]^T$