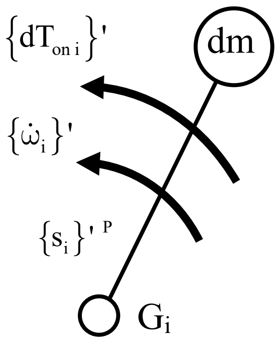
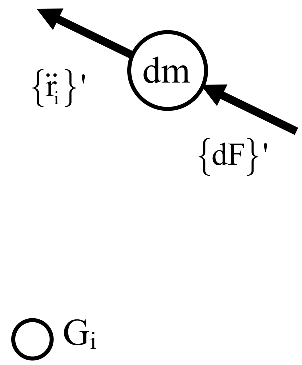
Three-Dimensional Mass Moment of Inertia

centroidal torque $\left\{dT_{on\,i}\right\}'$ in local directions to cause angular acceleration $\{\dot{\omega}_i\}'$ in local directions of differential mass dm at local location $\{s_i\}'^{P}$





$\{\ddot{r}_i\}' = \{\dot{\omega}_i\}' \times \{s_i\}'^{P}$ $\qquad$ $\left\{dT_{on\,i}\right\}' = \{s_i\}'^{P} \times \{dF\}'$

$$\{dF\}' = dm\left(\{\dot{\omega}_i\}' \times \{s_i\}'^{P}\right) = -dm\left(\{s_i\}'^{P} \times \{\dot{\omega}_i\}'\right) = -dm\left([\tilde{s}_i]'^{P}\,\{\dot{\omega}_i\}'\right)$$

$$\left\{T_{on\,i}\right\}' = \int_m \left(\{s_i\}'^{P} \times \{dF\}'\right) = \int_m \left([\tilde{s}_i]'^{P}\,\{dF\}'\right) = -\int_m \left([\tilde{s}_i]'^{P}\,[\tilde{s}_i]'^{P}\,\{\dot{\omega}_i\}'\right) dm$$

$$\left\{T_{on\,i}\right\}' = [J_{Gi}]'\{\dot{\omega}_i\}'$$

$$[J_{Gi}]' = -\int_m \left([\tilde{s}_i]'^{P}\,[\tilde{s}_i]'^{P}\right) dm = \int_m \begin{bmatrix} \left(y_i'^{P}\right)^2 + \left(z_i'^{P}\right)^2 & -x_i'^{P} y_i'^{P} & -x_i'^{P} z_i'^{P} \\ -x_i'^{P} y_i'^{P} & \left(x_i'^{P}\right)^2 + \left(z_i'^{P}\right)^2 & -y_i'^{P} z_i'^{P} \\ -x_i'^{P} z_i'^{P} & -y_i'^{P} z_i'^{P} & \left(x_i'^{P}\right)^2 + \left(y_i'^{P}\right)^2 \end{bmatrix} dm$$

alternately $\quad -[\tilde{s}_i]'^{P}\,[\tilde{s}_i]'^{P} = \left(\{s_i\}'^{P}\right)^T \{s_i\}'^{P}\,[I_3] - \{s_i\}'^{P}\left(\{s_i\}'^{P}\right)^T$

$$[J_{Gi}]' = \begin{bmatrix} J'_{XX\,i} & J'_{XY\,i} & J'_{XZ\,i} \\ J'_{XY\,i} & J'_{YY\,i} & J'_{YZ\,i} \\ J'_{XZ\,i} & J'_{YZ\,i} & J'_{ZZ\,i} \end{bmatrix}$$

rotated into global directions $\qquad \{s_i\}^{P} = [A_i]\{s_i\}'^{P}$

$$[J_{Gi}] = [A_i][J_{Gi}]'[A_i]^T$$

about global origin $\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$

$$[J_i] = [A_i][J_{Gi}]'[A_i]^T + m_i\left(\{r_i\}^T\{r_i\}[I_3] - \{r_i\}\{r_i\}^T\right)$$

about local origin j $\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$ $\{r_j\}^P = \{r_j\} + [A_j]\{s_j\}'^P$ $\{\rho\} = \{r_j\} - \{r_i\}$

$$[J_{Oi}]' = [A_j]^T\left([A_i][J_{Gi}]'[A_i]^T + m_i\left(\{\rho\}^T\{\rho\}[I_3] - \{\rho\}\{\rho\}^T\right)\right)[A_j]$$

$$[J_{Gi}]' = \begin{bmatrix} J'_{XX\,i} & J'_{XY\,i} & J'_{XZ\,i} \\ J'_{XY\,i} & J'_{YY\,i} & J'_{YZ\,i} \\ J'_{XZ\,i} & J'_{YZ\,i} & J'_{ZZ\,i} \end{bmatrix}$$

$J'_{XY} = 0$ $J'_{XZ} = 0$ $J'_{YZ} = 0$ for symmetric objects and about principal axes

eigenvalues are principal components along diagonal $[D]$

eigenvectors are unit vectors for principal directions in columns of $[V]$

$$[J_G] = [V][D][V]^T$$

angular momentum in local directions

$$\{H_i\}' = \begin{bmatrix} J'_{XX\,i} & J'_{XY\,i} & J'_{XZ\,i} \\ J'_{XY\,i} & J'_{YY\,i} & J'_{YZ\,i} \\ J'_{XZ\,i} & J'_{YZ\,i} & J'_{ZZ\,i} \end{bmatrix} \begin{Bmatrix} \omega_{Xi}' \\ \omega_{Yi}' \\ \omega_{Zi}' \end{Bmatrix} \qquad [H_i]' = [J_{Gi}]'\{\omega_i\}'$$

angular momentum in global directions

$$\{H_i\} = [J_{Gi}]\{\omega_i\} = [A_i]\{H_i\}'$$

rotational kinetic energy

$$K_\omega = \tfrac{1}{2}\{\omega_i\}'^T[J_{Gi}]'\{\omega_i\}' = \tfrac{1}{2}\{\omega_i\}^T[J_{Gi}]\{\omega_i\}$$

Rotational equations of motion in local coordinates

$$\{H_i\} = [J_{Gi}]\{\omega_i\}$$

$$\{\dot{H}_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$\{\dot{H}_i\} = [\dot{J}_{Gi}]\{\omega_i\} + [J_{Gi}]\{\dot{\omega}_i\}$$

$$[J_{Gi}] = [A_i][J_{Gi}]'[A_i]^T$$

$$[\dot{J}_{Gi}] = [\dot{A}_i][J_{Gi}]'[A_i]^T + [A_i][J_{Gi}]'[\dot{A}_i]^T \qquad [\dot{A}_i] = [\tilde{\omega}_i][A_i] = [A_i][\tilde{\omega}_i]'$$

$$[\dot{J}_{Gi}] = [A_i][\tilde{\omega}_i]'[J_{Gi}]'[A_i]^T + [A_i][J_{Gi}]'[A_i]^T[\tilde{\omega}_i]^T$$

$$\{\dot{H}_i\} = [J_{Gi}]\{\dot{\omega}_i\} + [\dot{J}_{Gi}]\{\omega_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[J_{Gi}]\{\dot{\omega}_i\} + \left([A_i][\tilde{\omega}_i]'[J_{Gi}]'[A_i]^T + [A_i][J_{Gi}]'[A_i]^T[\tilde{\omega}_i]^T\right)\{\omega_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[J_{Gi}]\{\dot{\omega}_i\} + [A_i][\tilde{\omega}_i]'[J_{Gi}]'[A_i]^T\{\omega_i\} - [A_i][J_{Gi}]'[A_i]^T[\tilde{\omega}_i]\{\omega_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[J_{Gi}]\{\dot{\omega}_i\} + [A_i][\tilde{\omega}_i]'[J_{Gi}]'[A_i]^T\{\omega_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[A_i][J_{Gi}]'[A_i]^T\{\dot{\omega}_i\} + [A_i][\tilde{\omega}_i]'[J_{Gi}]'[A_i]^T\{\omega_i\} = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[A_i][J_{Gi}]'\{\dot{\omega}_i\}' + [A_i][\tilde{\omega}_i]'[J_{Gi}]'\{\omega_i\}' = \sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[J_{Gi}]'\{\dot{\omega}_i\}' + [\tilde{\omega}_i]'[J_{Gi}]'\{\omega_i\}' = [A_i]^T\sum\left(\{T_{on\ i}\}_{ALL}\right)$$

$$[J_{Gi}]'\{\dot{\omega}_i\}' + [\tilde{\omega}_i]'[J_{Gi}]'\{\omega_i\}' = \sum\left(\{T_{on\ i}\}'_{ALL}\right)$$

$$[J_{Gi}]'\{\dot{\omega}_i\}' = \sum\left(\{T_{on\ i}\}'_{ALL}\right) - [\tilde{\omega}_i]'[J_{Gi}]'\{\omega_i\}'$$

$$m_i\{\ddot{r}_i\} = \sum\left(\{F_{on\ i}\}_{ALL}\right)$$