Three-Dimensional Dynamics

<u>Must use centroidal coordinate frames</u>

$$\left[\Phi_r\right] = \begin{bmatrix} \left[\Phi_{r2}\right]_{KINEMATIC} & \left[\Phi_{r3}\right]_{KINEMATIC} & \cdots \\ \left[\Phi_{r2}\right]_{DRIVER} & \left[\Phi_{r3}\right]_{DRIVER} & \cdots \end{bmatrix} \quad \text{for} \quad \{\ddot{r}\} = \begin{Bmatrix} \{\ddot{r}_2\} \\ \{\ddot{r}_3\} \\ \vdots \end{Bmatrix}$$

$$\left[\Phi_{\pi'}\right] = \begin{bmatrix} \left[\Phi_{\pi'2}\right]_{KINEMATIC} & \left[\Phi_{\pi'3}\right]_{KINEMATIC} & \cdots \\ \left[\Phi_{\pi'2}\right]_{DRIVER} & \left[\Phi_{\pi'3}\right]_{DRIVER} & \cdots \end{bmatrix} \quad \text{for} \quad \{\omega\}' = \begin{Bmatrix} \{\omega_2\}' \\ \{\omega_3\}' \\ \vdots \end{Bmatrix} \qquad \{\dot{\omega}\}' = \begin{Bmatrix} \{\dot{\omega}_2\}' \\ \{\dot{\omega}_3\}' \\ \vdots \end{Bmatrix}$$

$$\left[\left[\Phi_r\right] \quad \left[\Phi_{\pi'}\right]\right] \begin{Bmatrix} \{\dot{r}\} \\ \{\omega\}' \end{Bmatrix} = \{\nu\} \qquad \left[\left[\Phi_r\right] \quad \left[\Phi_{\pi'}\right]\right] \begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \{\gamma\}$$

$$\left[*_{pi}\right] = 2\left[*_{\pi'i}\right]\left[G_i\right] \qquad \left[*_{\pi'i}\right] = \tfrac{1}{2}\left[*_{pi}\right]\left[G_i\right]^T$$

$$\left[M_i\right] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix}$$

$$\left[M\right] = \begin{bmatrix} \left[M_2\right] & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & \left[M_3\right] & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}$$

$$\left[J\right]' = \begin{bmatrix} \left[J_{G2}\right]' & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & \left[J_{G3}\right]' & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}$$

$$\left[\tilde{\omega}\right]' = \begin{bmatrix} \left[\tilde{\omega}_2\right]' & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & \left[\tilde{\omega}_3\right]' & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}$$

$$\{F_{on\ i}\}_{ALL} \qquad \{F_{on\ i}\}_{APPLIED} \qquad \{F_{on\ i}\}_{CONSTRAINT}$$

$$\{F\}_{APPLIED} = \begin{Bmatrix} \sum\left(\{F_{on\ 2}\}_{APPLIED}\right) \\ \sum\left(\{F_{on\ 3}\}_{APPLIED}\right) \\ \vdots \end{Bmatrix}$$

$$\{T\}'_{APPLIED} = \begin{Bmatrix} \sum\left(\{T_{on\ 2}\}'_{APPLIED}\right) \\ \sum\left(\{T_{on\ 3}\}'_{APPLIED}\right) \\ \vdots \end{Bmatrix}$$

Equations of motion (EOM)

$$[M]\{\ddot{r}\} + [\Phi_r]^T\{\lambda\} = \{F\}_{APPLIED}$$

$$[J]'\{\dot{\omega}\}' + [\Phi_{\pi'}]^T\{\lambda\} = \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}'$$

$$\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix} \begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \{\gamma\} \qquad \text{or} \qquad [\Phi_q]\{\ddot{q}\} = \{\gamma\}$$

$$\underset{3nb\ x\ 1}{\{\ddot{r}\}} \quad \underset{3nb\ x\ 3nb}{[M]} \quad \underset{3nb\ x\ 1}{\{F\}_{APPLIED}} \quad \underset{nc\ x\ 3nb}{[\Phi_r]} \quad \underset{nc\ x\ 1}{\{\lambda\}}$$

$$\underset{3nb\ x\ 1}{\{\dot{\omega}\}'} \quad \underset{3nb\ x\ 3nb}{[J]'} \quad \underset{3nb\ x\ 1}{\{T\}'_{APPLIED}} \quad \underset{3nb\ x\ 1}{[\tilde{\omega}]'[J]'\{\omega\}'} \quad \underset{nc\ x\ 3nb}{[\Phi_{\pi'}]} \quad \underset{nc\ x\ 1}{\{\gamma\}}$$

nb = number of moving bodies
nk = number of kinematic constraints
nd = number of driver constraints
nc = total number of constraints (nc = nk + nd)

Inverse dynamics – kinematically driven

solve kinematics $\begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}^{-1}\{\gamma\}$

$\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}$ must have full rank nc = 6nb

compute constraint forces $\{\lambda\} = \left(\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}^T\right)^{-1} \begin{Bmatrix} \{F\}_{APPLIED} - [M]\{\ddot{r}\} \\ \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}' - [J]'\{\dot{\omega}\}' \end{Bmatrix}$

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix} \qquad \{\lambda\} = \begin{Bmatrix} \{\lambda\}_{KINEMATIC} \\ \{\lambda\}_{DRIVER} \end{Bmatrix}$$

Statics

$\{\ddot{r}\} = \{0\}$ and $\{\omega\}' = \{0\}$ and $\{\dot{\omega}\}' = \{0\}$

$$\{\lambda\} = \left(\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}^T\right)^{-1} \begin{Bmatrix} \{F\}_{APPLIED} \\ \{T\}'_{APPLIED} \end{Bmatrix}$$

Inverse dynamics – simultaneous EOM matrix

$$\begin{bmatrix} \underset{3nb \times 3nb}{[M]} & 0_{3nb \times 3nb} & \underset{3nb \times nc}{[\Phi_r]^T} \\ 0_{3nb \times 3nb} & \underset{3nb \times 3nb}{[J]'} & \underset{3nb \times nc}{[\Phi_{\pi'}]^T} \\ \underset{nc \times 3nb}{[\Phi_r]} & \underset{nc \times 3nb}{[\Phi_{\pi'}]} & 0_{nc \times nc} \end{bmatrix} \begin{Bmatrix} \underset{3nb \times 1}{\{\ddot{r}\}} \\ \underset{3nb \times 1}{\{\dot{\omega}\}'} \\ \underset{nc \times 1}{\{\lambda\}} \end{Bmatrix} = \begin{Bmatrix} \underset{3nb \times 1}{\{F\}_{APPLIED}} \\ \underset{3nb \times 1}{\{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}'} \\ \underset{nc \times 1}{\{\gamma\}} \end{Bmatrix}$$

$$[EOM] = \underset{(6nb+nc) \times (6nb+nc)}{\begin{bmatrix} [M] & 0_{3nb \times 3nb} & [\Phi_r]^T \\ 0_{3nb \times 3nb} & [J]' & [\Phi_{\pi'}]^T \\ [\Phi_r] & [\Phi_{\pi'}] & 0_{nc \times nc} \end{bmatrix}}$$

$$\begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & 0_{3nb \times 3nb} & [\Phi_r]^T \\ 0_{3nb \times 3nb} & [J]' & [\Phi_{\pi'}]^T \\ [\Phi_r] & [\Phi_{\pi'}] & 0_{nc \times nc} \end{bmatrix}^{-1} \begin{Bmatrix} \{F\}_{APPLIED} \\ \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}' \\ \{\gamma\} \end{Bmatrix}$$

Lagrange multipliers for specific constraints

$$\{F_{on\ i}\}_{CONSTRAINT} = -[\Phi_{r\ i}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$$

$$\{T_{on\ i}\}'_{CONSTRAINT} = \left([\tilde{s}_i]'^P\,[A_i]^T\,[\Phi_{r\ i}]^T_{CONSTRAINT} - [\Phi_{\pi'\ i}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$$

$$\{F_{on\ j}\}_{CONSTRAINT} = -[\Phi_{r\ j}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$$

$$\{T_{on\ j}\}'_{CONSTRAINT} = \left([\tilde{s}_j]'^P\,[A_j]^T\,[\Phi_{r\ j}]^T_{CONSTRAINT} - [\Phi_{\pi'\ j}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$$

local joint definition frame at P

$$\{s_i\}'^A = \{s_i\}'^P + [C_i]^P\{s_i\}''^A$$

$$\{F_{on\ i}\}'' = \left([C_i]^P\right)^T[A_i]^T\{F_{on\ i}\}$$

$$\{T_{on\ i}\}'' = \left([C_i]^P\right)^T\{T_{on\ i}\}'$$

Spherical

$$\{\Phi\}_{SPH} = \{r_j\}^P - \{r_i\}^P = \{0_{3x1}\}$$

$$\{F_{on\ i}\}_{SPH} = \{\lambda\}_{SPH}$$

$$\{T_{on\ i}\}'_{SPH} = \{0_{3x1}\}$$

$$\{F_{on\ j}\}_{SPH} = -\{\lambda\}_{SPH}$$

$$\{T_{on\ j}\}'_{SPH} = \{0_{3x1}\}$$

Double spherical

$$\Phi_{SPH\_SPH} = \{d_{ij}\}^T\{d_{ij}\} - C^2 = 0 \qquad \text{for} \quad \{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$$

$$\{F_{on\ i}\}_{CONSTRAINT} = 2\{d_{ij}\}\lambda_{SPH\_SPH}$$

$$\{T_{on\ i}\}'_{CONSTRAINT} = \{0_{3x1}\}$$

$$\{F_{on\ j}\}_{CONSTRAINT} = -2\{d_{ij}\}\lambda_{SPH\_SPH}$$

$$\{T_{on\ j}\}'_{CONSTRAINT} = \{0_{3x1}\}$$

Dot-1

$$\Phi_{DOT\_1} = \{a_i\}^T\{a_j\} = 0 \qquad \text{for} \quad \{a_i\} = \{r_i\}^Q - \{r_i\}^P \quad \text{and} \quad \{a_j\} = \{r_j\}^Q - \{r_j\}^P$$

$$\{F_{on\ i}\}_{DOT\_1} = \{0_{1x3}\}$$

$$\{T_{on\ i}\}'_{DOT\_1} = -[\tilde{a}_i]'[A_i]^T[A_j]\{a_j\}'\lambda_{DOT\_1}$$

$$\{F_{on\ j}\}_{DOT\_1} = \{0_{1x3}\}$$

$$\{T_{on\ j}\}'_{DOT\_1} = -[\tilde{a}_j]'[A_j]^T[A_i]\{a_i\}'\lambda_{DOT\_1}$$

Dot-2

$$\Phi_{DOT\_2} = \{a_i\}^T\{d_{ij}\} = 0 \qquad \text{for} \quad \{d_{ij}\} = \{r_j\}^P - \{r_i\}^P \quad \text{and} \quad \{a_i\} = \{r_i\}^Q - \{r_i\}^P$$

$$\{F_{on\ i}\}_{DOT\_2} = \{a_i\}\lambda_{DOT\_2}$$

$$\{T_{on\ i}\}'_{DOT\_2} = -[\tilde{a}_i]'[A_i]^T\{d_{ij}\}\lambda_{DOT\_2}$$

$$\{F_{on\ j}\}_{DOT\_2} = -\{a_i\}\lambda_{DOT\_2}$$

$$\{T_{on\ j}\}'_{DOT\_2} = 0$$

Forward dynamics - dynamically driven

nc < 6nb

$\lfloor [\Phi_r] \quad \lfloor \Phi_{\pi'} \rfloor \rfloor$ does not have full row rank

given $\{r\}$ $\{p\}$ $\{\dot{r}\}$ $\{\omega\}'$ $\{F\}_{APPLIED}$ $\{T\}'_{APPLIED}$ $[M]$ $[J]'$

compute $[\Phi_r]$ $\lfloor \Phi_{\pi'} \rfloor$ $\{\gamma\}$

$$\begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & 0_{3nb\,x\,3nb} & [\Phi_r]^T \\ 0_{3nb\,x\,3nb} & [J]' & \lfloor \Phi_{\pi'} \rfloor^T \\ [\Phi_r] & \lfloor \Phi_{\pi'} \rfloor & 0_{nc\,x\,nc} \end{bmatrix}^{-1} \begin{Bmatrix} \{F\}_{APPLIED} \\ \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}' \\ \{\gamma\} \end{Bmatrix}$$

must use forward time integration of $\{\ddot{r}\}$ $\{\dot{\omega}\}'$ to get $\{r\}$ $\{p\}$ $\{\dot{r}\}$ $\{\omega\}'$ at next time step

$$\underset{13nb}{\{y\}} = \begin{Bmatrix} \underset{3nb}{\{r\}} \\ \underset{4nb}{\{p\}} \\ \underset{3nb}{\{\dot{r}\}} \\ \underset{3nb}{\{\omega\}'} \end{Bmatrix} \qquad \underset{13nb}{\{\dot{y}\}} = \begin{Bmatrix} \underset{3nb}{\{\dot{r}\}} \\ \underset{4nb}{\{\dot{p}\}} \\ \underset{3nb}{\{\ddot{r}\}} \\ \underset{3nb}{\{\dot{\omega}\}'} \end{Bmatrix} \qquad \underset{4nb}{\{\dot{p}\}} = \tfrac{1}{2}[G]^T \underset{3nb}{\{\omega\}'} \qquad \underset{3nbx\,4nb}{[G]} = \begin{bmatrix} [G_2] & 0_{3x4} & 0_{3x4} \\ 0_{3x4} & [G_3] & 0_{3x4} \\ 0_{3x4} & 0_{3x4} & \ddots \end{bmatrix}$$

EOM using Euler parameters

$$\begin{bmatrix} [M] & 0_{3nb\ x\ 4nb} & [\Phi_r]^T & 0_{3nb\ x\ nb} \\ 0_{4nb\ x\ 3nb} & 4[G]^T[J]'[G] & [\Phi_p]^T & ([\Phi_p]_E)^T \\ [\Phi_r] & [\Phi_p] & 0_{nc\ x\ nc} & 0_{nc\ x\ nb} \\ 0_{nb\ x\ 3nb} & [\Phi_p]_E & 0_{nb\ x\ nc} & 0_{nb\ x\ nb} \end{bmatrix} \begin{Bmatrix} \{\ddot{r}\} \\ \{\ddot{p}\} \\ \{\lambda\} \\ \{\lambda\}_E \end{Bmatrix} = \begin{Bmatrix} \{F\}_{APPLIED} \\ 2[G]^T\{T\}'_{APPLIED} + 8[\dot{G}]^T[J]'[\dot{G}]\{p\} \\ \{\gamma\} \\ \{\gamma\}_E \end{Bmatrix}$$

$$\{p\} = \begin{Bmatrix} \{p_2\} \\ \{p_3\} \\ \vdots \end{Bmatrix} \qquad [G] = \begin{bmatrix} [G_2] & 0_{3x4} & 0_{3x4} \\ 0_{3x4} & [G_3] & 0_{3x4} \\ 0_{3x4} & 0_{3x4} & \ddots \end{bmatrix}$$

$$[\Phi_p]_E = 2\begin{bmatrix} \{p_2\}^T & 0_{1x4} & 0_{1x4} \\ 0_{1x4} & \{p_2\}^T & 0_{1x4} \\ 0_{1x4} & 0_{1x4} & \ddots \end{bmatrix} \qquad \{\gamma\}_E = -2\begin{Bmatrix} \{\dot{p}_2\}^T\{\dot{p}_2\} \\ \{\dot{p}_3\}^T\{\dot{p}_3\} \\ \vdots \end{Bmatrix}$$

Euler parameter

$$\{\Phi\}_{EUL} = \{p_i\}^T\{p_i\} - 1 = 0$$

$$[\Phi_{ri}]_{EUL} = \{0_{1x3}\}$$

$$[\Phi_{\pi' i}]_{EUL} = \{p_i\}^T[G_i]^T = ([G_i]\{p_i\})^T = \{0_{1x3}\}$$

$$[\Phi_{pi}]_{EUL} = 2\{p_i\}^T$$

$$\{F_{on\ i}\}_{EUL} = \{0_{3x1}\}$$

$$\{T_{on\ i}\}'_{EUL} = \{0_{3x1}\}$$

$$\{F_{on\ j}\}_{EUL} = \{0_{3x1}\}$$

$$\{T_{on\ j}\}'_{EUL} = \{0_{3x1}\}$$