# Newton-Raphson Algorithm

x=3.75

$$x_{k+1} = x_k - f_k / (\partial f / \partial x)_k$$

**Example**: $x^2 - 4x + 1 = 0$ **Use:** $f(x) = x^2 - 4x + 1$ $\partial f / \partial x = 2x - 4$

| x | f(x) |
|---|---|
| 0 | 1 |
| 1 | -2 |
| 2 | -3 |
| 3 | -2 |
| 4 | 1 |
| 5 | 6 |

guess $x_k$
iteration $k$
$x=5$
evaluate $f(x)=6$
evaluate $\frac{\partial f}{\partial x}=8$

$$\Delta x = -\frac{f}{\left(\frac{\partial f}{\partial x}\right)}$$

$x_{k+1} = x_k + \Delta x$

residual slope

| k | x | f(x) | ∂f/∂x | f / (∂f/∂x) |
|---|---|---|---|---|
| 1 | 5 | 6 | 6 | 1 |
| 2 | 4 | 1 | 4 | 0.25 |
| 3 | 3.75 | 0.0625 | 3.5 | 0.01786 |
| 4 | 3.7321 | 0.0003 | 3.4643 | 0.00009 |
| 5 | 3.7320 | | | |
| | | | | |
| 1 | 2.5 | -2.75 | 1 | -2.75 |
| 2 | 5.25 | 7.5625 | 6.5 | 1.1635 |
| 3 | 4.0865 | 1.3536 | 4.1731 | 0.3244 |
| 4 | 3.7622 | 0.1052 | 3.5243 | 0.0299 |
| 5 | 3.7323 | | | |
| | | | | |
| 1 | 1 | -2 | -2 | 1 |
| 2 | 0 | 1 | -4 | -0.25 |
| 3 | 0.25 | 0.0625 | -3.5 | -0.01786 |
| 4 | 0.2679 | 0.00017 | -3.4642 | -0.00005 |
| 5 | 0.2679 | | | |

# Newton-Raphson for Four Bar

**Given:** constants $r_1$ $r_2$ $r_3$ $r_4$ $\theta_1$ and variable $\theta_2$ **Find:** $\theta_3$ and $\theta_4$

**Subject to:**

constraints

$$\boxed{\begin{aligned} f_H &= r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1\cos\theta_1 = 0 \\ f_V &= r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 - r_1\sin\theta_1 = 0 \end{aligned}}$$

estimate $\{q\}_k$

row

**Use:** $\{q\} = \begin{Bmatrix}\theta_3\\ \theta_4\end{Bmatrix}$ generalized coordinates, $\{\Phi\} = \begin{Bmatrix} f_H \\ f_V\end{Bmatrix}$ constraint functions

evaluate $\{\Phi\}$ residuals

columns

new

**Taylor series about estimate $\{q\}_k$:**

$$\cancel{\begin{Bmatrix} f_H \\ f_V\end{Bmatrix}_{k+1}} \approx \begin{Bmatrix} f_H \\ f_V\end{Bmatrix}_k + \begin{bmatrix} \frac{\partial f_H}{\partial\theta_3} & \frac{\partial f_H}{\partial\theta_4} \\ \frac{\partial f_V}{\partial\theta_3} & \frac{\partial f_V}{\partial\theta_4}\end{bmatrix} \begin{Bmatrix}\Delta\theta_3 \\ \Delta\theta_4\end{Bmatrix}$$

hope zero

evaluate Jacobian $[\Phi_q]$

$$\{\Phi\}_{k+1} \approx \{\Phi\}_k + [\Phi_q]_k \{\Delta q\}_k \qquad [\Phi_q] = [\partial\Phi/\partial q] = \begin{bmatrix} \frac{\partial f_H}{\partial\theta_3} & \frac{\partial f_H}{\partial\theta_4} \\ \frac{\partial f_V}{\partial\theta_3} & \frac{\partial f_V}{\partial\theta_4}\end{bmatrix} = \boxed{\begin{bmatrix} -r_3\sin\theta_3 & r_4\sin\theta_4 \\ r_3\cos\theta_3 & -r_4\cos\theta_4\end{bmatrix}}$$

$\frac{\partial\Phi}{\partial q}$

**Desired constraint functions:** $\{\Phi\}_{k+1} = 0$ $\qquad \{\Delta q\} = -[\Phi_q]_k^{-1}\{\Phi\}_k$

hope

**Newton-Raphson equation:** $\{q\}_{k+1} = \{q\}_k - [\Phi_q]_k^{-1}\{\Phi\}_k$

**Example: $r_1$ = 90 cm, $r_2$ = 30 cm, $r_3$ = 60 cm, $r_4$ = 45 cm, $\theta_1$ = 0°, $\theta_2$ = 65°**

($\theta_3$ = 13.151°, $\theta_4$ = 114.827° by manual geometric solution)

| k | $\{q\} = \begin{Bmatrix}\theta_3\\ \theta_4\end{Bmatrix}$ [deg] | $\{\Phi\} = \begin{Bmatrix} f_H \\ f_V\end{Bmatrix}$ residual [cm] | $[\partial\Phi/\partial q]$ [cm/rad] | $[\partial\Phi/\partial q]^{-1}\{\Phi\}$ [rad] | $[\partial\Phi/\partial q]^{-1}\{\Phi\}$ [deg] |
|---|---|---|---|---|---|
| 1 | $\theta_3$ 0<br>$\theta_4$ 90 | -17.3215<br>-17.8108 | 0 45<br>60 0 | -0.2968<br>-0.3849 | -17.0080<br>-22.0544 |
| 2 | 17.0080<br>112.0544 | -3.0488<br>3.0323 | -17.5503 41.7073<br>57.3758 16.8969 | 0.0662<br>-0.0453 | 3.7916<br>-2.5927 |
| 3 | 13.2164<br>114.6471 | -0.1444<br>0.0068 | -13.7178 40.9002<br>58.4108 18.7663 | 0.0011<br>-0.0032 | 0.0647<br>-0.1806 |
| 4 | 13.1517<br>114.8277 | -0.00013<br>0.00019 | -13.6518 40.8409<br>58.4263 18.8951 | 0.0000039<br>-0.0000018 | 0.00023<br>-0.00011 |
| 5 | 13.1515<br>114.8278 | $10^{-14}$<br>$10^{-14}$ | det $[\Phi_q]$ =<br>-2644.1 | | |

$\gamma = \theta_4 - \theta_3 = 101.6763°$ $\qquad -r_3 r_4 \sin\gamma = -2644.1$

1) max 10 iterations
2) check max(abs{Φ}) < ∈
3) check max(abs(updates)) < ∈

if $\theta_4$ is the driver $\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$ $[\Phi_q] = \begin{bmatrix} -r_2 \sin\theta_2 & -r_3 \sin\theta_3 \\ r_2 \cos\theta_2 & r_3 \cos\theta_3 \end{bmatrix}$ $\det[\Phi_q] = r_2 r_3 \sin(\theta_3 - \theta_2)$

if $\theta_3$ is the driver $\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_4 \end{Bmatrix}$ $[\Phi_q] = \begin{bmatrix} -r_2 \sin\theta_2 & r_4 \sin\theta_4 \\ r_2 \cos\theta_2 & -r_4 \cos\theta_4 \end{bmatrix}$ $\det[\Phi_q] = r_2 r_4 \sin(\theta_2 - \theta_4)$

know $\theta_2$, find $\theta_3$ $\theta_4$

## Numerical Partial Derivatives

1) use current generalized coordinates $\{q\}$

2) evaluate constraint functions $\{\Phi\}$

3) perturb one (and only one) $q_j$ by a small value $\varepsilon$ $\qquad q_j^* = q_j + \varepsilon$

4) evaluate perturbed constraint functions $\{\Phi^*\}$

5) compute partial derivatives with respect to that $q_j$ $\qquad \dfrac{\partial\{\Phi\}}{\partial q_j} = \dfrac{\{\Phi^*\} - \{\Phi\}}{\varepsilon}$

6) column j of Jacobian $\left[\partial\Phi / \partial q\right] = \dfrac{\partial\{\Phi\}}{\partial q_j}$

7) remember to reset and use the original value for $q_j$

8) repeat steps 3) through 7) for all j

**Example: $r_1$ = 90 cm, $r_2$ = 30 cm, $r_3$ = 60 cm, $r_4$ = 45 cm, $\theta_1$ = 0°, $\theta_2$ = 65°**

$$\{\Phi\} = \begin{Bmatrix} f_H \\ f_V \end{Bmatrix} = \begin{Bmatrix} r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1\cos\theta_1 \\ r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 - r_1\sin\theta_1 \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 90 \end{Bmatrix}\text{deg} = \begin{Bmatrix} 0 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.8108 \end{Bmatrix}\text{cm} \qquad \varepsilon = 0.001\text{ rad}$$

$$\{q\} = \begin{Bmatrix} 0 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.8108 \end{Bmatrix}\text{cm} \qquad \varepsilon = 0.001\text{ rad}$$

$$\{q\}^* = \begin{Bmatrix} 0.001 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.7508 \end{Bmatrix}\text{cm} \qquad \frac{\partial\{\Phi\}}{\partial q_1} = \begin{Bmatrix} -0.0300 \\ 60.0000 \end{Bmatrix}\text{cm/rad}$$

$$\{q\}^* = \begin{Bmatrix} 0 \\ 1.5718 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.2765 \\ -17.7508 \end{Bmatrix}\text{cm} \qquad \frac{\partial\{\Phi\}}{\partial q_1} = \begin{Bmatrix} 45.0000 \\ 0.0225 \end{Bmatrix}\text{cm/rad}$$

$$\left[\partial\Phi / \partial q\right] \approx \begin{bmatrix} -0.0300 & 45.0000 \\ 60.0000 & 0.0225 \end{bmatrix}\text{cm/rad}$$

$$\begin{bmatrix} 0 & 45 \\ 60 & 0 \end{bmatrix}$$

```
% test_jac.m - evaluate Jacobian by numerical partial derivatives
%   used for ME 581 web cutter
% HJSIII, 20.02.19

% hold estimates for generalized coordinates
nq = length(q);
qhold = q;

% evaluate constraints
wc_phi

% hold constraints
phold = PHI;

% perturb one coordinate at a time
for iq = 1:nq,
  q = qhold;
  q(iq) = q(iq) + 0.01;

% change in constraints caused by coordinate perturbation is
% approximately equal to partial derivative
  wc_phi
  jtest(:,iq) = ( PHI - phold ) / 0.01;
end

% reset coordinates and constraints
q = qhold;
wc_phi

% bottom - test_jac
```