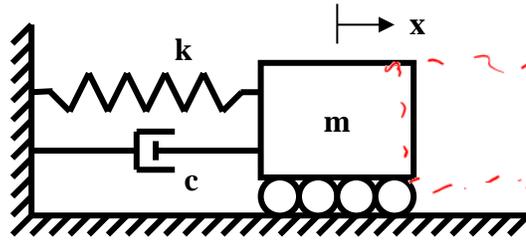


*Newton Raphson*  
*{q}*  
*velocity*  
*{v}*

### State Space Model For Spring-Mass-Damper



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x}$$

$$\{q\} = x \quad \{\dot{q}\} = \dot{x} \quad \{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

use states  $\{y\}$  to compute derivative of states  $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\{\dot{y}\} = [A]\{y\} \quad \{\dot{y}\} = f(\{y\})$$

*[A] ≠ rotation matrix*

**assumed solution**  $\{y\} = \begin{Bmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{Bmatrix}$   $\{\dot{y}\} = \begin{Bmatrix} \lambda a_1 e^{\lambda t} \\ \lambda a_2 e^{\lambda t} \end{Bmatrix} = \lambda \{y\}$

$$\{\dot{y}\} = [A]\{y\} \quad \lambda \{y\} = [A]\{y\} \quad [A]\{y\} - \lambda \{y\} = \{0\}$$

$$([A] - \lambda[I_2])\{y\} = \{0\} \quad \det([A] - \lambda[I_2]) = 0 \quad \lambda \text{ is an eigenvalue, NOT } \{\lambda\} \text{ from EOM}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & (-\frac{c}{m} - \lambda) \end{bmatrix} = 0 \quad \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

**for c = 0,**  $\lambda = \sqrt{-\frac{k}{m}} = i\sqrt{\frac{k}{m}} = i\omega_n$

$$\omega_n = \sqrt{\frac{k}{m}}$$

**critical damping**  $\left(\frac{c_{CR}}{2m}\right)^2 - \frac{k}{m} = 0 \quad c_{CR} = 2m\omega_n \quad \zeta = \frac{c}{c_{CR}} = \frac{c}{2m\omega_n}$

**underdamped**  $\zeta < 1 \quad \lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

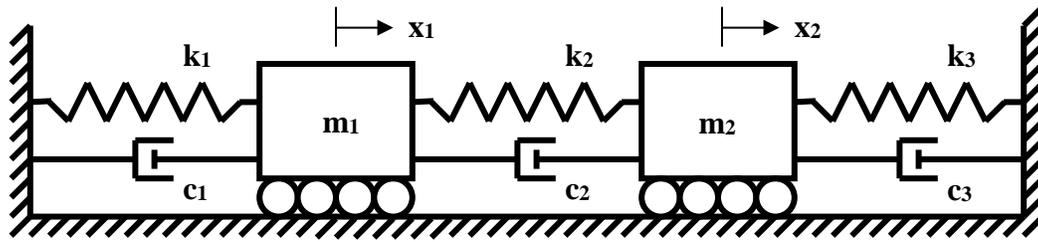
*rearrange to solve for x*

*first order ODE*

*{y} state*

*{pos}*  
*{vel}*

### State Space Model For Double Spring-Mass-Damper

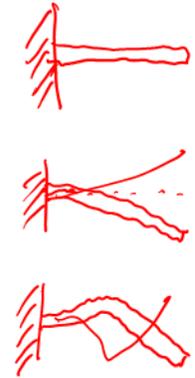


$$m_1 \ddot{x}_1 = -k_1 x_1 - c_1 \dot{x}_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2)$$

$$m_2 \ddot{x}_2 = -k_3 x_2 - c_3 \dot{x}_2 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1)$$

$$\{q\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad \{\dot{q}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \quad \{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

*pos*      *vel*      *state*



use states  $\{y\}$  to compute derivative of states  $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1+k_2}{m_1}\right) & +\frac{k_2}{m_1} & -\left(\frac{c_1+c_2}{m_1}\right) & +\frac{c_2}{m_1} \\ +\frac{k_2}{m_2} & -\left(\frac{k_2+k_3}{m_2}\right) & +\frac{c_2}{m_2} & -\left(\frac{c_2+c_3}{m_2}\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = [A]\{y\}$$

*state*

$\{\dot{y}\} = [A]\{y\}$        $\{\dot{y}\} = f(\{y\})$

assumed solution  $\{y\} = \{a_i e^{\lambda t}\}$        $\{\dot{y}\} = \lambda \{y\}$

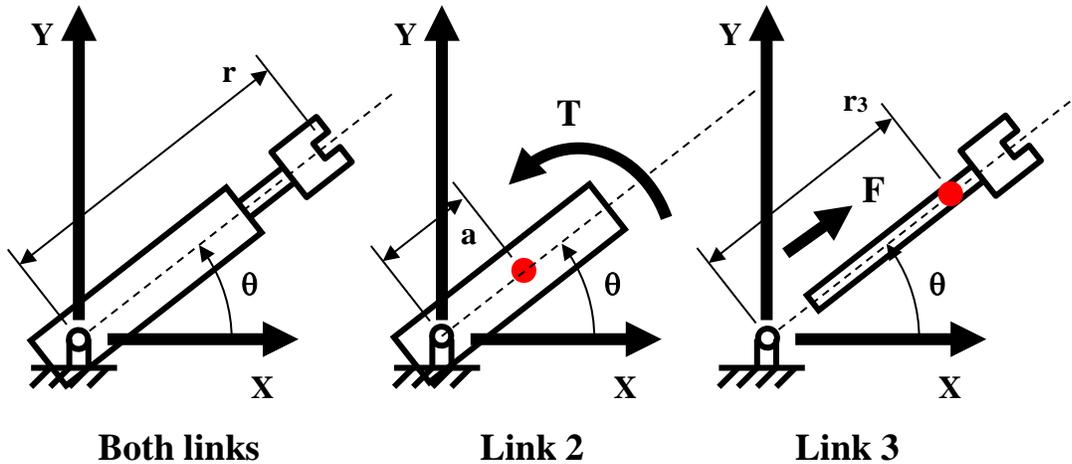
for  $c_1 = c_2 = c_3 = 0$ ,

$$a = (m_1 k_2 + m_1 k_3 + m_2 k_1 + m_2 k_2) / (m_1 m_2), \quad b = (k_1 k_2 + k_1 k_3 + k_2 k_3) / (m_1 m_2)$$

$$\lambda = i \sqrt{\frac{a}{2} \left( 1 \pm \sqrt{1 - \frac{4b}{a^2}} \right)}$$

*eigen values of [A]*

## State Space Model for Cylindrical Coordinate Manipulator



Main body link 2 - Shaft and end-effector link 3

Mass centers at  $a$  and  $r_3$  from waist rotation axis,  $a = \text{constant}$ ,  $r_3 = \text{variable}$

Masses  $m_2$  and  $m_3$  - centroidal mass moments of inertia  $J_2$  and  $J_3$

$\theta$  CCW from positive x axis -  $a$  and  $r_3$  radial from rotation axis

$T$  is rotary actuator torque of ground on body 2 about waist measured CCW positive

$F$  is radial actuator force of body 2 on body 3 measured positive outward

Gravity  $g$  acts along negative y axis

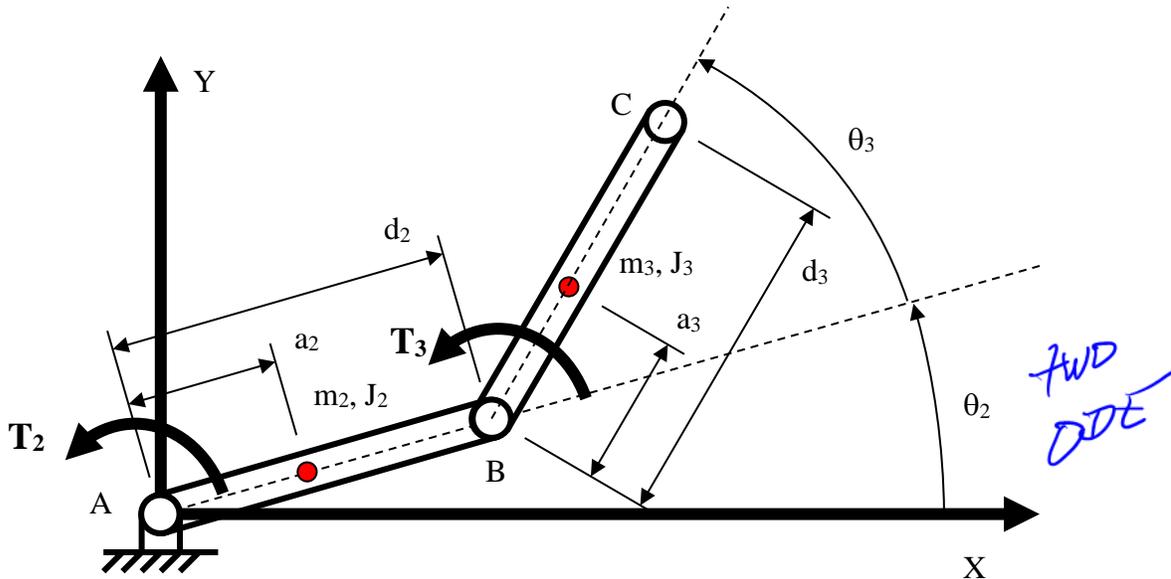
$$\begin{bmatrix} m_2 a^2 + m_3 r_3^2 + J_2 + J_3 & 0 \\ 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{r}_3 \end{Bmatrix} = \begin{Bmatrix} T - 2m_3 r_3 \dot{r}_3 \dot{\theta} - g(m_2 a + m_3 r_3) \cos \theta \\ F + m_3 r_3 \dot{\theta}^2 - m_3 g \sin \theta \end{Bmatrix}$$

$$\begin{aligned} \{q\} &= \begin{Bmatrix} \theta \\ r_3 \end{Bmatrix} & \{\dot{q}\} &= \begin{Bmatrix} \dot{\theta} \\ \dot{r}_3 \end{Bmatrix} & \{y\} &= \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} \theta \\ r_3 \\ \dot{\theta} \\ \dot{r}_3 \end{Bmatrix} \\ \text{pos} & & \text{vel} & & \text{state} & \end{aligned}$$

use states  $\{y\}$  to compute derivative of states  $\dot{y} = f(\{y\})$

$$\underbrace{\{\dot{y}\}} = \underbrace{\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}} = \begin{Bmatrix} \dot{\theta} \\ \dot{r}_3 \\ \ddot{\theta} \\ \ddot{r}_3 \end{Bmatrix} = \begin{Bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ r_3 \\ \dot{\theta} \\ \dot{r}_3 \end{Bmatrix} \\ \begin{bmatrix} m_2 a^2 + m_3 r_3^2 + J_2 + J_3 & 0 \\ 0 & m_3 \end{bmatrix}^{-1} \begin{Bmatrix} T - 2m_3 r_3 \dot{r}_3 \dot{\theta} - g(m_2 a + m_3 r_3) \cos \theta \\ F + m_3 r_3 \dot{\theta}^2 - m_3 g \sin \theta \end{Bmatrix} \end{Bmatrix}$$

## State Space Model for Two Link Anthropomorphic Manipulator (Double Pendulum)



Two solid rigid bars with revolute joints A and B

Lengths  $d_2$  and  $d_3$  - mass centers at  $a_2$  and  $a_3$  from proximal ends

Masses  $m_2$  and  $m_3$  - centroidal mass moments of inertia  $J_2$  and  $J_3$

$\theta_2$  CCW from positive x axis

$T_2$  is torque of ground on bar 2 about pin A, CCW positive

$\theta_3$  CCW from centerline of bar 2

$T_3$  is torque of bar 2 on bar 3 about pin B, CCW positive

Gravity  $g$  acts along negative y axis

$$J_B = m_3 a_3^2 + J_3$$

$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2m_3 d_2 a_3 \cos \theta_3$$

$$C = J_B + m_3 d_2 a_3 \cos \theta_3$$

$$D = m_3 d_2 a_3 \sin \theta_3$$

$$G_2 = (m_2 a_2 + m_3 d_2) g \cos \theta_2$$

$$G_3 = m_3 a_3 g \cos (\theta_2 + \theta_3)$$

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left( \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

$$\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$\{\dot{q}\} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}$$

$$\{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}$$

gave  $\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$   
 use forward dynamics  
 got  $\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$

use states  $\{y\}$  to compute derivative of states  $\{\dot{y}\} = f(\{y\})$

$$\dot{\{y\}} = \begin{Bmatrix} \dot{\{q\}} \\ \dot{\{q\}} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \\ \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left( \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right) \end{Bmatrix}$$

$$\{\dot{y}\} = f(\{y\})$$

$$\{y\} = \begin{Bmatrix} \{pos\} \\ \{vel\} \end{Bmatrix}$$

$$\{\dot{y}\} = \begin{Bmatrix} \{vel\} \\ \{acc\} \end{Bmatrix}$$

$$f(\{y\})$$

~~$$[A]\{y\}$$~~

*pretend*

### Numerically Evaluate Linear State Matrix

assume  $\{\dot{y}\} \approx [A_{\text{LINEAR}}]\{y\}$  using ns number of states and n number of time samples

$$\begin{bmatrix} \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t1} & \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t2} & \dots & \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{tn} \end{bmatrix} \approx [A_{\text{LINEAR}}] \begin{bmatrix} \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t1} & \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t2} & \dots & \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{tn} \end{bmatrix}$$

*4x4   4x4   4x4   10 steps*

$$\begin{bmatrix} \dot{Y} \\ \vdots \\ \dot{Y}_{ns} \end{bmatrix} \approx [A_{\text{LINEAR}}] \begin{bmatrix} Y \\ \vdots \\ Y_{ns} \end{bmatrix}$$

$\begin{matrix} ns \times n & & ns \times ns & & ns \times n \end{matrix}$

*remember old values for  $\{y\}$  =  $\begin{Bmatrix} \{y_1\} \\ \{y_2\} \\ \{y_3\} \end{Bmatrix}$*   
*remember old values for  $\{y\}$  =  $\begin{Bmatrix} \{y_1\} \\ \{y_2\} \\ \{y_3\} \\ \{y_4\} \end{Bmatrix}$*

Note that samples need not be computed using fixed h, and do not need to be continuous.

*n = # of step that were remembered*

for  $n = ns$

$$[A_{\text{LINEAR}}] \approx [\dot{Y}][Y]^{-1} = \begin{bmatrix} \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t1} & \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t2} & \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t3} & \left\{ \begin{matrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{matrix} \right\}_{t4} \end{bmatrix} \begin{bmatrix} \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t1} & \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t2} & \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t3} & \left\{ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{matrix} \right\}_{t4} \end{bmatrix}^{-1}$$

for  $n \geq ns$

$$[A_{\text{LINEAR}}] \approx [\dot{Y}][Y]^T ([Y][Y]^T)^{-1}$$

~~*n < ns*~~

~~*...*~~

extract  $\{\lambda\}$  from  $[A_{\text{LINEAR}}]$

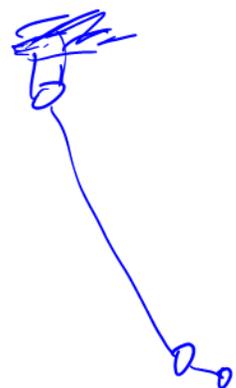
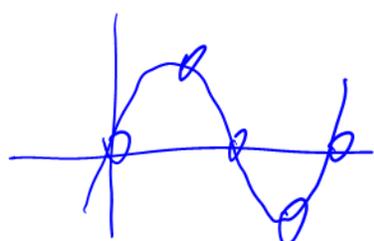
$$\{f_n\} = \frac{\text{abs}\{\lambda\}}{2\pi}$$

$$h < \frac{1}{2 \max\{f_n\}} \quad \text{prefer} \quad h < \frac{1}{10 \max\{f_n\}}$$

*cycles/sec*

*Nyquist*

*Sommer*



*recommendation for h*