

Three-Dimensional Coordinate Transformations

$$\underline{\{r_i\}^P} = \{r_i\} + [A_i] \{s_i\}'^P \quad \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}^P = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \begin{bmatrix} a_{i11} & a_{i12} & a_{i13} \\ a_{i21} & a_{i22} & a_{i23} \\ a_{i31} & a_{i32} & a_{i33} \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}'^P$$

$$\{s_i\}^P = [A_i] \{s_i\}'^P \quad \{s_i\}'^P = [A_i]^T \{s_i\}^P$$

[A] matrices are orthonormal $[A]^{-1} = [A]^T$

- all columns are unit vectors
- all columns are mutually orthogonal
- all rows are unit vectors
- all rows are mutually orthogonal
- $\det [A] = +1$

Single rotations

$$[A_{\theta_x}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_x & -S\theta_x \\ 0 & S\theta_x & C\theta_x \end{bmatrix}$$

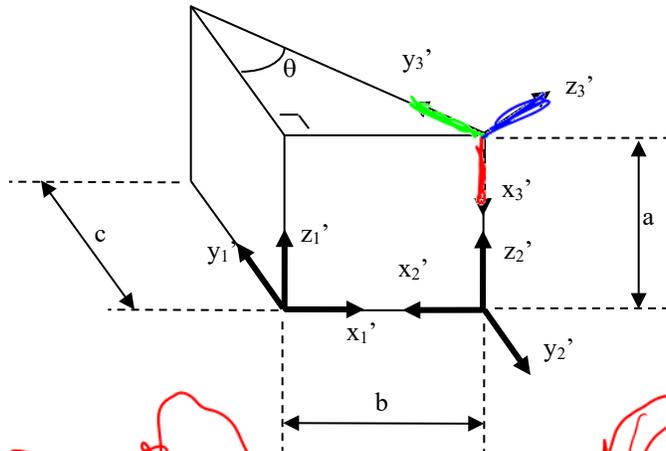
$$[A_{\theta_y}] = \begin{bmatrix} C\theta_y & 0 & S\theta_y \\ 0 & 1 & 0 \\ -S\theta_y & 0 & C\theta_y \end{bmatrix}$$

$$[A_{\theta_z}] = \begin{bmatrix} C\theta_z & -S\theta_z & 0 \\ S\theta_z & C\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Handwritten diagram illustrating a rotation around the z-axis. It shows a coordinate system with axes x, y, and z. A rotation matrix is written in red: $\begin{bmatrix} C\theta_z & -S\theta_z & 0 \\ S\theta_z & C\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A green arrow indicates the rotation of the x and y axes in the xy-plane. The original axes are labeled x, y, z and the rotated axes are labeled x', y', z'.

Coordinate Frames at the Corners of a Wedge

Adapted from Introduction to Robotics, J.J. Craig, Addison-Wesley, 1989



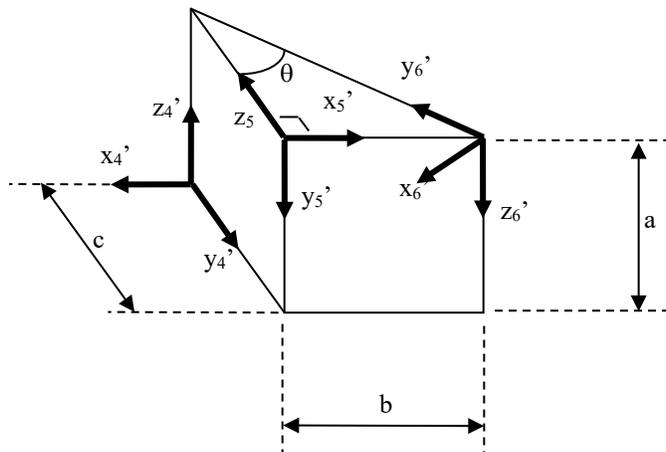
~~$$\begin{bmatrix} 0 & -\cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 & \cos\theta \\ -1 & 0 & 0 & 0 \end{bmatrix}$$~~

$s^2\theta - (-c\theta)$

$s^2\theta + c^2\theta = 1$

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}$$

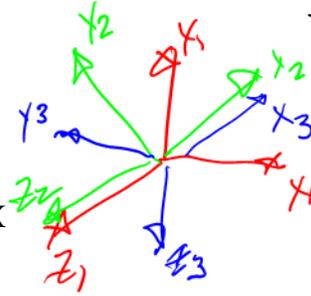
$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \\ a \end{Bmatrix} + \begin{bmatrix} 0 & -S\theta & C\theta \\ 0 & C\theta & S\theta \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix}$$



$$\begin{Bmatrix} x_4 \\ y_4 \\ z_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ c \\ a \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} x_5 \\ y_5 \\ z_5 \end{Bmatrix}$$

$$\begin{Bmatrix} x_4 \\ y_4 \\ z_4 \end{Bmatrix} = \begin{Bmatrix} -b \\ c \\ a \end{Bmatrix} + \begin{bmatrix} C\theta & S\theta & 0 \\ S\theta & -C\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_6 \\ y_6 \\ z_6 \end{Bmatrix}$$

Euler Angles



Euler sequences – ZXZ (original), ZYZ, YXY, YZY, XYX, XZX

ZXZ sequence (θ_1 about global z - θ_2 about intermediate x - θ_3 about local z)

$$[A_{\theta_z}][A_{\theta_x}][A_{\theta_z}] = \begin{matrix} \theta_z & & \theta_x & & \theta_z \\ \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_2 & -S\theta_2 \\ 0 & S\theta_2 & C\theta_2 \end{bmatrix} & \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ S\theta_3 & C\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

forward solution $\rightarrow [A]$
 $\theta_z \theta_x \theta_z$

$$[A_{\theta_z}][A_{\theta_x}][A_{\theta_z}] = \begin{bmatrix} -S\theta_1 C\theta_2 S\theta_3 + C\theta_1 C\theta_3 & -S\theta_1 C\theta_2 C\theta_3 - C\theta_1 S\theta_3 & \underline{S\theta_1 S\theta_2} \\ C\theta_1 C\theta_2 S\theta_3 + S\theta_1 C\theta_3 & C\theta_1 C\theta_2 C\theta_3 - S\theta_1 S\theta_3 & \underline{-C\theta_1 S\theta_2} \\ \underline{S\theta_2 S\theta_3} & \underline{S\theta_2 C\theta_3} & \underline{C\theta_2} \end{bmatrix}$$

inverse solution $[A] \rightarrow \theta_z \theta_x \theta_z$
 fails because gimbal lock

$\theta_2 = \arccos(a_{33})$ $\theta_1 = \arctan(-a_{13}/a_{23})$ $\theta_3 = \arctan(a_{31}/a_{32})$ fails for $\theta_2 = 0$ and π

ZYZ sequence

YXY sequence

YZY sequence

XYX sequence

XZX sequence

Cardan-Bryant-Tait sequences – XYZ, XZY, YXZ, YZX, ZXY, ZYX**XYZ sequence** (θ_x about global x - θ_y about intermediate y - θ_z about local z)

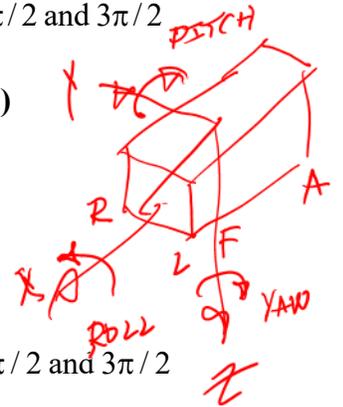
$$[A_{0x}][A_{0y}][A_{0z}] = \begin{bmatrix} C\theta_y C\theta_z & -C\theta_y S\theta_z & S\theta_y \\ S\theta_x S\theta_y C\theta_z + C\theta_x S\theta_z & -S\theta_x S\theta_y S\theta_z + C\theta_x C\theta_z & -S\theta_x C\theta_y \\ -C\theta_x S\theta_y C\theta_z + S\theta_x S\theta_z & C\theta_x S\theta_y S\theta_z + S\theta_x C\theta_z & C\theta_x C\theta_y \end{bmatrix}$$



$$\theta_y = a \sin(a_{13}) \quad \theta_z = a \tan 2(-a_{12}/a_{11}) \quad \theta_x = a \tan 2(-a_{23}/a_{33}) \quad \text{fails for } \theta_y = \pi/2 \text{ and } 3\pi/2$$

ZYX sequence (θ_z about global z - θ_y about intermediate y - θ_x about local x)

$$[A_{0z}][A_{0y}][A_{0x}] = \begin{bmatrix} C\theta_y C\theta_z & S\theta_x S\theta_y C\theta_z - C\theta_x S\theta_z & C\theta_x S\theta_y C\theta_z + S\theta_x S\theta_z \\ C\theta_y S\theta_z & S\theta_x S\theta_y S\theta_z + C\theta_x C\theta_z & C\theta_x S\theta_y S\theta_z - S\theta_x C\theta_z \\ -S\theta_y & S\theta_x C\theta_y & C\theta_x C\theta_y \end{bmatrix}$$



$$\theta_y = a \sin(-a_{31}) \quad \theta_x = a \tan 2(a_{32}/a_{33}) \quad \theta_z = a \tan 2(a_{21}/a_{11}) \quad \text{fails for } \theta_y = \pi/2 \text{ and } 3\pi/2$$

XZY sequence (θ_x about global x - θ_z about intermediate z - θ_y about local y)

$$[A_{0x}][A_{0z}][A_{0y}] = \begin{bmatrix} C\theta_y C\theta_z & -S\theta_z & S\theta_y C\theta_z \\ C\theta_x C\theta_y S\theta_z + S\theta_x S\theta_y & C\theta_x C\theta_z & C\theta_x S\theta_y S\theta_z - S\theta_x C\theta_y \\ S\theta_x C\theta_y S\theta_z - C\theta_x S\theta_y & S\theta_x C\theta_z & S\theta_x S\theta_y S\theta_z + C\theta_x C\theta_y \end{bmatrix}$$

$$\theta_z = a \sin(-a_{12}) \quad \theta_x = a \tan 2(a_{32}/a_{22}) \quad \theta_y = a \tan 2(a_{13}/a_{11}) \quad \text{fails for } \theta_z = \pi/2 \text{ and } 3\pi/2$$

YZX sequence**YXZ sequence****ZXY sequence**

Derivatives of Cardan-Bryant-Tait angles – ZYX**ZYX sequence (θ_z about global z - θ_y about intermediate y - θ_x about local x)**

$$[A_{0z}][A_{0y}][A_{0x}] = \begin{bmatrix} C\theta_Y C\theta_Z & C\theta_Z S\theta_Y S\theta_X - S\theta_Z C\theta_X & C\theta_X S\theta_Y C\theta_Z + S\theta_X S\theta_Z \\ C\theta_Y S\theta_Z & S\theta_X S\theta_Y S\theta_Z + C\theta_X C\theta_Z & C\theta_X S\theta_Y S\theta_Z - C\theta_Z S\theta_X \\ -S\theta_Y & S\theta_X C\theta_Y & C\theta_X C\theta_Y \end{bmatrix}$$

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix} + [A_{0z}] \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix} + [A_{0z}][A_{0y}] \begin{Bmatrix} \dot{\theta}_X \\ 0 \\ 0 \end{Bmatrix}$$

$$[A_{0z}] \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_Z & -S\theta_Z & 0 \\ S\theta_Z & C\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix}$$

$$[A_{0z}][A_{0y}] \begin{Bmatrix} \dot{\theta}_X \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_Z & -S\theta_Z & 0 \\ S\theta_Z & C\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_Y & 0 & S\theta_Y \\ 0 & 1 & 0 \\ -S\theta_Y & 0 & C\theta_Y \end{bmatrix} \begin{Bmatrix} \dot{\theta}_X \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_Z C\theta_Y & -S\theta_Z & C\theta_Z S\theta_Y \\ S\theta_Z C\theta_Y & C\theta_Z & S\theta_Z S\theta_Y \\ -S\theta_Y & 0 & C\theta_Y \end{bmatrix} \begin{Bmatrix} \dot{\theta}_X \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{bmatrix} C\theta_Z C\theta_Y & -S\theta_Z & 0 \\ S\theta_Z C\theta_Y & C\theta_Z & 0 \\ -S\theta_Y & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_X \\ \dot{\theta}_Y \\ \dot{\theta}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_X \\ \dot{\theta}_Y \\ \dot{\theta}_z \end{Bmatrix} = \frac{1}{C\theta_Y} \begin{bmatrix} C\theta_Z & S\theta_Z & 0 \\ -C\theta_Y S\theta_Z & C\theta_Y C\theta_Z & 0 \\ S\theta_Y C\theta_Z & S\theta_Y S\theta_Z & C\theta_Y \end{bmatrix} \begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} \quad \text{fails for } \theta_Y = \pi/2 \text{ and } 3\pi/2$$

$$\{\omega\}' = [A]\{\omega\}$$

$$\{\omega\}' = [A]^T \{\omega\} = [A_{0x}]^T [A_{0y}]^T [A_{0z}]^T \{\omega\}$$

$$\begin{Bmatrix} \omega_X' \\ \omega_Y' \\ \omega_Z' \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_X \\ 0 \\ 0 \end{Bmatrix} + [A_{0x}]^T \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix} + [A_{0x}]^T [A_{0y}]^T \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix}$$

$$[A_{\theta_x}]^T \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_x & S\theta_x \\ 0 & -S\theta_x & C\theta_x \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix}$$

$$[A_{\theta_x}]^T [A_{\theta_Y}]^T \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_x & S\theta_x \\ 0 & -S\theta_x & C\theta_x \end{bmatrix} \begin{bmatrix} C\theta_Y & 0 & -S\theta_Y \\ 0 & 1 & 0 \\ S\theta_Y & 0 & C\theta_Y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix} = \begin{bmatrix} C\theta_Y & 0 & -S\theta_Y \\ S\theta_x S\theta_Y & C\theta_x & S\theta_x C\theta_Y \\ C\theta_x S\theta_Y & -S\theta_x & C\theta_x C\theta_Y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -S\theta_Y \\ 0 & C\theta_x & S\theta_x C\theta_Y \\ 0 & -S\theta_x & C\theta_x C\theta_Y \end{bmatrix} \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_Y \\ \dot{\theta}_z \end{Bmatrix}$$

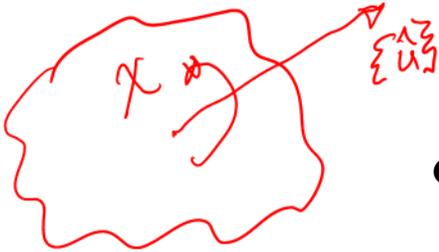
$$\begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_Y \\ \dot{\theta}_z \end{Bmatrix} = \frac{1}{C\theta_Y} \begin{bmatrix} C\theta_Y & S\theta_x S\theta_Y & C\theta_x S\theta_Y \\ 0 & C\theta_x C\theta_Y & -S\theta_x C\theta_Y \\ 0 & S\theta_x & C\theta_x \end{bmatrix} \begin{Bmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{Bmatrix}$$

fails for $\theta_Y = \pi/2$ and $3\pi/2$

$\text{trace } (u^2+v^2+w^2)V\chi + 3C\chi$

$1 - C\chi + 3C\chi$

$\text{trace } 2C\chi + 1$



Chasles' Angle and Euler Parameters

Rotation χ about unit direction $\{\hat{u}\}$

$$[A] = \begin{bmatrix} u^2V\chi + C\chi & uvV\chi - wS\chi & uwV\chi + vS\chi \\ uvV\chi + wS\chi & v^2V\chi + C\chi & vwV\chi - uS\chi \\ uwV\chi - vS\chi & vwV\chi + uS\chi & w^2V\chi + C\chi \end{bmatrix}$$

$C\chi = (\text{tr}[A] - 1) / 2$

$\{\hat{u}\} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$= \frac{1}{2S\chi} \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$ *2uSχ*

fails for $\chi = 0$ and π

Euler parameters (unit quaternion)

$\{\mathbf{p}\} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} C \frac{\chi}{2} \\ uS \frac{\chi}{2} \\ vS \frac{\chi}{2} \\ wS \frac{\chi}{2} \end{bmatrix}$

$\{\mathbf{p}\}^T \{\mathbf{p}\} = e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$

$[A] = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$

$e_0^2 = (\text{tr}[A] + 1) / 4$

$\{\mathbf{e}\} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{4e_0} \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$

fails for $e_0 = 0$ at $\chi = \pi$

given find $\{\hat{u}\}$ and χ

given $\{\hat{u}\}$ and χ

$V\chi = 1 - C\chi$

given $\Sigma[A]$

find $\{\hat{u}\}$ and χ

given $\epsilon p^3 \rightarrow \Sigma[A]$

given $\Sigma[A] \rightarrow \epsilon p^3$

$\{A\}$

A_1

A_2

A_3

A_4

$$[\mathbf{E}] = \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_0 & -\mathbf{e}_3 & \mathbf{e}_2 \\ -\mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_0 & -\mathbf{e}_1 \\ -\mathbf{e}_3 & -\mathbf{e}_2 & \mathbf{e}_1 & \mathbf{e}_0 \end{bmatrix} \quad [\mathbf{G}] = \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ -\mathbf{e}_2 & -\mathbf{e}_3 & \mathbf{e}_0 & \mathbf{e}_1 \\ -\mathbf{e}_3 & \mathbf{e}_2 & -\mathbf{e}_1 & \mathbf{e}_0 \end{bmatrix}$$

$$[\mathbf{A}] = [\mathbf{E}][\mathbf{G}]^T = \begin{bmatrix} \mathbf{e}_0^2 + \mathbf{e}_1^2 - \mathbf{e}_2^2 - \mathbf{e}_3^2 & 2(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_0\mathbf{e}_3) & 2(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_0\mathbf{e}_2) \\ 2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_0\mathbf{e}_3) & \mathbf{e}_0^2 - \mathbf{e}_1^2 + \mathbf{e}_2^2 - \mathbf{e}_3^2 & 2(\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_0\mathbf{e}_1) \\ 2(\mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_0\mathbf{e}_2) & 2(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_0\mathbf{e}_1) & \mathbf{e}_0^2 - \mathbf{e}_1^2 - \mathbf{e}_2^2 + \mathbf{e}_3^2 \end{bmatrix}$$

$$[\mathbf{E}][\mathbf{E}]^T = [\mathbf{G}][\mathbf{G}]^T = [\mathbf{I}_3]$$

$$[\mathbf{E}]^T[\mathbf{E}] = [\mathbf{G}]^T[\mathbf{G}] = [\mathbf{I}_4] - \{\mathbf{p}\}\{\mathbf{p}\}^T$$

$$[\mathbf{E}]\{\mathbf{p}\} = [\mathbf{G}]\{\mathbf{p}\} = \{0\}$$

Velocity

$$\{\omega\} = [\mathbf{A}]\{\omega\}'$$

$$\{\omega\} = 2[\mathbf{E}]\{\dot{\mathbf{p}}\} = -2[\dot{\mathbf{E}}]\{\mathbf{p}\}$$

$$\{\omega\}' = [\mathbf{A}]^T \{\omega\}$$

$$\{\omega\}' = 2[\mathbf{G}]\{\dot{\mathbf{p}}\} = -2[\dot{\mathbf{G}}]\{\mathbf{p}\}$$

$$\{\dot{\mathbf{p}}\} = \frac{1}{2}[\mathbf{E}]^T \{\omega\} = \frac{1}{2}[\mathbf{G}]^T \{\omega\}'$$

$$\{\mathbf{p}\}^T \{\dot{\mathbf{p}}\} = \{\dot{\mathbf{p}}\}^T \{\mathbf{p}\} = 0$$

$$[\dot{\omega}] = [\dot{\mathbf{A}}][\mathbf{A}]^T = -2[\mathbf{E}][\dot{\mathbf{E}}]^T = 2[\dot{\mathbf{E}}][\mathbf{E}]^T$$

$$[\dot{\omega}]' = [\mathbf{A}]^T [\dot{\mathbf{A}}] = 2[\mathbf{G}][\dot{\mathbf{G}}]^T = -2[\dot{\mathbf{G}}][\mathbf{G}]^T$$

$$[\dot{\mathbf{A}}] = [\dot{\omega}][\mathbf{A}] = [\mathbf{A}][\dot{\omega}]' = 2[\mathbf{E}][\dot{\mathbf{G}}]^T = 2[\dot{\mathbf{E}}][\mathbf{G}]^T$$

Acceleration

$$\{\dot{\omega}\} = [A]\{\dot{\omega}\}'$$

$$\{\dot{\omega}\} = 2[E]\{\ddot{p}\}$$

$$\{\dot{\omega}\}' = [A]^T \{\dot{\omega}\}$$

$$\{\dot{\omega}\}' = 2[G]\{\ddot{p}\}$$

$$\{\ddot{p}\} = \frac{1}{2}[E]^T \{\dot{\omega}\} + \frac{1}{2}[\dot{E}]^T \{\omega\} = \frac{1}{2}[E]^T \{\dot{\omega}\} - \frac{1}{4}\{p\}(\{\omega\}^T \{\omega\})$$

$$\{\ddot{p}\} = \frac{1}{2}[G]^T \{\dot{\omega}\}' + \frac{1}{2}[\dot{G}]^T \{\omega\}' = \frac{1}{2}[G]^T \{\dot{\omega}\}' - \frac{1}{4}\{p\}'(\{\omega\}'^T \{\omega\}')$$

$$\{\dot{p}\}^T \{\dot{p}\} = \frac{1}{4}\{\omega\}^T \{\omega\} = \frac{1}{4}\{\omega\}'^T \{\omega\}' = \frac{1}{4}\omega^2$$

$$[\dot{E}]\{\dot{p}\} = [\dot{G}]\{\dot{p}\} = 0$$

$$[\ddot{A}] = 2[\dot{E}][\dot{G}]^T + 2[E][\dot{G}]^T = 2[\ddot{E}][G]^T + 2[\dot{E}][\dot{G}]^T$$

$$[E][\ddot{G}]^T = [\ddot{E}][G]^T$$

Jerk

$$\{\ddot{\omega}\} = [A]\{\dot{\omega}\}' + [A][\tilde{\omega}]\{\dot{\omega}\}'$$

$$\{\ddot{\omega}\} = 2[E]\{\ddot{p}\} + 2[\dot{E}]\{\dot{p}\} = 2[E]\{\ddot{p}\} + \frac{1}{2}[\tilde{\omega}]\{\dot{\omega}\} + \frac{1}{4}\{\omega\}\{\omega\}^T\{\omega\}$$

$$\{\dot{\omega}\}' = [A]^T\{\ddot{\omega}\} - [A]^T[\tilde{\omega}]\{\dot{\omega}\}'$$

$$\{\dot{\omega}\}' = 2[G]\{\ddot{p}\} + 2[\dot{G}]\{\dot{p}\} = 2[G]\{\ddot{p}\} - \frac{1}{2}[\tilde{\omega}]\{\dot{\omega}\}' + \frac{1}{4}\{\omega\}'\{\omega\}'^T\{\omega\}'$$

$$\begin{aligned}\{\ddot{p}\} &= \frac{1}{2}[E]^T\{\ddot{\omega}\} + [\dot{E}]^T\{\dot{\omega}\}' + \frac{1}{2}[\ddot{E}]^T\{\omega\} \\ &= \frac{1}{2}[E]^T\left(\{\ddot{\omega}\} - \frac{1}{2}[\tilde{\omega}]\{\dot{\omega}\}' - \frac{1}{2}\{\omega\}\{\omega\}^T\{\omega\}\right) - \frac{3}{4}\{\dot{p}\}\{\omega\}^T\{\dot{\omega}\}'\end{aligned}$$

$$\begin{aligned}\{\ddot{p}\} &= \frac{1}{2}[G]^T\{\dot{\omega}\}' + [\dot{G}]^T\{\dot{\omega}\}' + \frac{1}{2}[\ddot{G}]^T\{\omega\}' \\ &= \frac{1}{2}[G]^T\left(\{\dot{\omega}\}' + \frac{1}{2}[\tilde{\omega}]\{\dot{\omega}\}' - \frac{1}{4}\{\omega\}'\{\omega\}'^T\{\omega\}'\right) - \frac{3}{4}\{\dot{p}\}\{\omega\}'^T\{\dot{\omega}\}'\end{aligned}$$

$$[\dot{G}][G]^T \neq 0 \quad [\dot{G}][\dot{G}]^T \neq 0 \quad [\dot{G}]\{\dot{p}\} \neq 0 \quad [G]\{\dot{p}\} \neq 0 \quad [\dot{G}]\{\dot{p}\} + [\ddot{G}]\{\dot{p}\} = 0$$

$$[\ddot{A}] = 2[E][\ddot{G}]^T + 4[\dot{E}][\dot{G}]^T + 2[\ddot{E}][\dot{G}]^T = 2[\ddot{E}][G]^T + 4[\dot{E}][\dot{G}]^T + 2[\ddot{E}][\ddot{G}]^T$$

$$[E][\ddot{G}]^T + [\dot{E}][\dot{G}]^T = [\ddot{E}][G]^T + [\ddot{E}][\dot{G}]^T$$

Snap

$$\{\ddot{\omega}\}' = 2[\mathbf{G}]\{\ddot{\mathbf{p}}\} + 2[\dot{\mathbf{G}}]\{\dot{\mathbf{p}}\} = 2[\mathbf{G}]\{\ddot{\mathbf{p}}\} - \frac{1}{2}[\dot{\tilde{\omega}}]\{\dot{\omega}\}' + \frac{1}{4}\{\omega\}'\{\omega\}'^T\{\omega\}'$$

$$\{\ddot{\omega}\}' = 2[\mathbf{G}]\{\ddot{\mathbf{p}}\} + 4[\dot{\mathbf{G}}]\{\dot{\mathbf{p}}\} + 2[\ddot{\mathbf{G}}]\{\mathbf{p}\}$$

$$\{\ddot{\mathbf{p}}\} = \frac{1}{2}[\mathbf{G}]^T\{\ddot{\omega}\}' + [\dot{\mathbf{G}}]^T\{\dot{\omega}\}' + \frac{1}{2}[\ddot{\mathbf{G}}]^T\{\omega\}'$$

$$\{\ddot{\mathbf{p}}\} = \frac{1}{2}[\mathbf{G}]^T\{\ddot{\omega}\}' + \frac{3}{2}[\dot{\mathbf{G}}]^T\{\dot{\omega}\}' + \frac{3}{2}[\ddot{\mathbf{G}}]^T\{\omega\}' + \frac{1}{2}[\ddot{\mathbf{G}}]^T\{\omega\}'$$

Numerical partial derivatives of rotation matrices with respect to Euler parameters can produce different results

$$[A] = 2 \begin{bmatrix} \mathbf{e}_0^2 + \mathbf{e}_1^2 - \frac{1}{2} & \mathbf{e}_1 \mathbf{e}_2 - \mathbf{e}_0 \mathbf{e}_3 & \mathbf{e}_1 \mathbf{e}_3 + \mathbf{e}_0 \mathbf{e}_2 \\ \mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_0 \mathbf{e}_3 & \mathbf{e}_0^2 + \mathbf{e}_2^2 - \frac{1}{2} & \mathbf{e}_2 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_1 \\ \mathbf{e}_1 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_2 & \mathbf{e}_2 \mathbf{e}_3 + \mathbf{e}_0 \mathbf{e}_1 & \mathbf{e}_0^2 + \mathbf{e}_3^2 - \frac{1}{2} \end{bmatrix} \quad \text{numerical } \frac{\partial \mathbf{a}_{11}}{\partial \mathbf{e}_3} = 0$$

$$[A] = [E][G]^T = \begin{bmatrix} \mathbf{e}_0^2 + \mathbf{e}_1^2 - \mathbf{e}_2^2 - \mathbf{e}_3^2 & 2(\mathbf{e}_1 \mathbf{e}_2 - \mathbf{e}_0 \mathbf{e}_3) & 2(\mathbf{e}_1 \mathbf{e}_3 + \mathbf{e}_0 \mathbf{e}_2) \\ 2(\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_0 \mathbf{e}_3) & \mathbf{e}_0^2 - \mathbf{e}_1^2 + \mathbf{e}_2^2 - \mathbf{e}_3^2 & 2(\mathbf{e}_2 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_1) \\ 2(\mathbf{e}_1 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_2) & 2(\mathbf{e}_2 \mathbf{e}_3 + \mathbf{e}_0 \mathbf{e}_1) & \mathbf{e}_0^2 - \mathbf{e}_1^2 - \mathbf{e}_2^2 + \mathbf{e}_3^2 \end{bmatrix} \quad \text{numerical } \frac{\partial \mathbf{a}_{11}}{\partial \mathbf{e}_3} = -2\mathbf{e}_3$$

Rodriguez Parameters

$$\{\mathbf{R}\} = \tan\left(\frac{\chi}{2}\right) \{\hat{\mathbf{u}}\}$$