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Monitoring Delayed Systems Using the Smith Predictor

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***Abstract*— Delays can be a critical property of control systems, since they introduce a frequency-proportional phase difference. In some applications, it is desirable to speed up the dynamic behavior of such systems. In this paper, a control will be provided that couples a delayed system that has a nonlinear state space function and linear input/output functions with a non-delayed model using a Smith Predictor, and criteria will be determined for the system to provide the same output signal as the model. In an ideal case, it will therefore be possible to monitor the model instead of the system. It will be proven that the given structure induces a synchronized state and a measure for the suitability of the control for a monitoring problem will be introduced. The results are then validated based on an exemplary system.**

I. INTRODUCTION

Synchronization and synchronizability are current topics in physics and mathematics. [1]–[12] Synchronization is, in this context, understood as identical solutions of systems at a certain time, and denoted as an universal concept of nonlinear and chaotic sciences [13], because it allows a simplified analysis of complex systems. These methods have been used in few excellent papers in control theory, [8], [9] yet provide a mathematical framework that can coveniently be expanded to a variety of control-theoretical problems. For the expansion of synchronization theory to systems with time delay, a sophisticated method of linear matrix inequalities has been developed. [5]–[12] In this paper, the Lambert W function will be employed for the analysis of the delay differential equations. This approach follows the methods outlined in some seminal publications [14]–[19]. Thus, this paper will first outline both, the theory of synchronization and the Lambert W function, and subsequently employ both for a novel interpretation of the Smith Predicter, which will consequently be analyzed.

Throughout this paper, italic letters denote vectors or scalars, whereas bold letters denote matrices. The respective dimensions will be given explicitly. Operators and predefined functions will, though, be formatted non-italic. The main results of this text, as well as the main results of prior publications, are formulated in lemmata.

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A. Synchronization

In a popular understanding, synchronized behavior is understood as an identical behavior of N systems at the same time. In an engineering sense, these systems have indentical state vectors at the time they achieve synchronization.

Definition 1. N systems are said to synchronize, as

$$\|x_i - x_j\| = 0 \,\forall\, i,j = 1,2,\cdots,N.$$

where x_i,$x_j \in \mathbb{R}^n$ are the state vectors of systems i, j. The dynamic behavior of each of these systems is assumed to be given by

$$\dot{x}_i = f_i\left(x_i\right) + h_i\left(u_i\right) \;\forall\, i = 1,2,\cdots,N,$$

where $\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} \in \mathbb{R}^n$ is the time-derivative of $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ is the input vector of system i and $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $h_i : \mathbb{R}^m \to \mathbb{R}^n$ are vector-valued functions expressing the dependency of $\dot{x}_i$ from x_i and u_i. Since controls couple system output with system inputs by feedbacks or other control-theoretical structures, every input vector u_i among N can be rewritten as a function of the states of the other systens. Thus,

$$u_i = g_i\left(x_j,\cdots,x_N\right),$$

where $g_i : \mathbb{R}^n \to \mathbb{R}^m$ describes the dependence of the input of the respective i from the other systems. Now it is assumed, that controls couple its systems by summations, subtractions, and gains. Thus, the function g_i has to be a linear combination resulting from these operations, so that

$$g_i = \sum_{j=1}^{N} \gamma_{ij} x_j$$

holds, where $\gamma = K$ represents a gain, and $\gamma = \pm 1$ a summation or subtraction, respectively. Additionally assuming h_i to be a linear function $\boldsymbol{K}_i \in \mathbb{R}^{n\times m}$, and $f_i = f_j \;\forall\, i,j = 1,2,\cdots N$ and simplifying the dynamic representation by imposing that all system dynamics are identical apart from the control coupling between them, $\boldsymbol{K}_i = \boldsymbol{K}_j \;\forall\; i,j = 1,2,\cdots N$ holds. The system dynamics thus simplify to

$$\dot{x}_i = f\left(x_i\right) + \sum_{j=j}^{N} \gamma_{ij} \boldsymbol{K}\, x_j \;\forall\, i,j = 1,2,\cdots,N,$$

where $\boldsymbol{K}$ is hereafter referred to as the inner coupling matrix and all γ_{ij} build a matrix $\boldsymbol{\Gamma} = [\gamma_{ij}] \in \mathbb{R}^{N\times N}$ that is called the coupling configuration matrix. The purpose of this notation is the formulation of one instead of N differential equations, that is

$$\dot{\chi} = F\left(\chi\right) + \left[\boldsymbol{\Gamma} \otimes \boldsymbol{K}\right]\chi,$$

where $\chi = \left[x_1^{\mathrm{T}} x_2^{\mathrm{T}} \cdots x_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{Nn}$ and $F(\chi) = \left[f^{\mathrm{T}}(x_1) f^{\mathrm{T}}(x_2) \cdots f^{\mathrm{T}}(x_N)\right]^{\mathrm{T}}$.

Definition 2. $\chi = \left[x_1^{\mathrm{T}}\ x_2^{\mathrm{T}}\ \cdots\ x_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{Nn}$ is the all-state vector of N systems.

For the analysis of the nonlinear dynamics of the network, the application of the first variation ξ of χ is required.

Definition 3. The first variation ξ of χ is the limit in θ of

$$\xi = \int \left. \frac{\mathrm{d}\dot{\chi}(\chi + \theta p)}{\mathrm{d}\theta} \right|_{\theta=0} dt$$

with χ any solution to $\dot{\chi}$ and p a specific solution.

By this definition and the substitution of the special case $p = \xi$, the dynamical behaviour of $\dot{\chi}$ may be rewritten by using ξ [20], so that the network representation is denoted as

$$\dot{\xi} = \left[\boldsymbol{I}^N \otimes \frac{\partial f}{\partial x} + \boldsymbol{\Gamma} \otimes \frac{\partial h}{\partial x}\right] \xi,$$

or, shorter,

$$\dot{\xi} = \left[\boldsymbol{I}^N \otimes \mathrm{D}f + \boldsymbol{\Gamma} \otimes \boldsymbol{K}\right] \xi,$$

where D is a differential operator and $\boldsymbol{I}^N = \mathrm{diag}\{1, 1, \cdots, 1\} \in \mathbb{R}^N$. Since the unit matrix has the eigenvectors $[1\ \ 0 \cdots 0]^{\mathrm{T}}$,$[0\ \ 1 \cdots 0]^{\mathrm{T}}$,$[0\ \ 0 \cdots 1]^{\mathrm{T}}$, each of these modes describes only the inner dynamics of each system. The eigenvectors of $\boldsymbol{\Gamma}$ describe the combinations of their respective modes. For a synchronous behavior, $\boldsymbol{\Gamma}$ should have at least one eigenvector of the form $[1 \cdots 1]^{\mathrm{T}} \in \mathbb{R}^N$, that implies a uniform time solution of all systems. This eigenvector has to be the eigenvector of a purely imaginary latent root, so that the synchronized movement is not damped out, and all other eigenvectors have to correspond with latent roots that have a non-positive real part, so that these modes do not dominate the synchronous mode.

Lemma 1 ([1]). System χ is said to have at least one synchronized state when at least one eigenvector of $\boldsymbol{\Gamma}$ has the form $[1 \cdots 1]^{\mathrm{T}} \in \mathbb{R}^N$

Lemma 2 ([1]). System χ is said to be synchronizable, if the synchronized state is the eigenvector of a purely imaginary latent root of $\boldsymbol{\Gamma}$ and all other eigenvalues have non-positive real parts.

For proving the stability of all systems, it is now only necessary to compute the set of Lyapunov exponents.

Definition 4. The set of Lypanuov exponents $(L_1, \cdots, L_{Nn})$ is the set of elements of $L = [L_1 \cdots L_{Nn}]^{\mathrm{T}}$ satisfying

$$\xi \propto e^{\boldsymbol{I}^{Nn} L t} \xi_0.$$

Hence, positive Lyapunov exponents represent orbital divergence and low predictability due to chaos, whereas negative Lyapunov exponents represent orbital convergence, high predictability and decay of perturbations [21], each with respect to the initial values. Since the first variation ξ has already been computed, the set of Lyapunov exponents for one system can be found by substituting a coupling coefficient Ω for each block of $\boldsymbol{\Gamma}$, so that it scales the impact of $\boldsymbol{K}$ in the same way as the eigenvalues of $\boldsymbol{\Gamma}$ do. Thus,

$$L = \sigma(\mathrm{D}f + \Omega \boldsymbol{K}),$$

where σ is the set of latent roots.

Definition 5. The set of latent roots σ of a matrix $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is the set of solutions to $|\boldsymbol{M} - \boldsymbol{I}^n \lambda| = 0$.

Hence, the the Lyapunov exponent is a function of Ω and therefore a function of $\boldsymbol{\Gamma}$. Now, both conditions bound the stable synchronized region.

Lemma 3 ([22]). A system χ has a conditional stable synchronized region if its coupling configuration matrix $\boldsymbol{\Gamma}$ fulfills

$$\sigma(\boldsymbol{\Gamma}) \in \{\Omega \in \mathbb{C} \,|\max(L) < 0\}.$$

By that, the given coupling coefficients correspond to a stable trajectory in the state space, because these Ω induce a negative Lyapunov exponent.

B. The Lambert W Function

Definition 6. The Lambert W function is the solution in $W(z)$ to $z = We^W$.

Thus, z is not injective, so that W is multivalued.

Definition 7. The $k \in \mathbb{Z}$ sets of solutions to W are called the branches of W. The kth solution of W is denoted as W_k. W_0 is called the principal branch.

The Lambert W function can be used to find the infinite set of solutions in transcendental algebraic equations. Its application is especially useful in solving delay-differential equations of the form

$$\dot{x} = \boldsymbol{A}\, x(t - \tau) + \boldsymbol{B}\, x,$$

where $\tau > 0$ is the delay and $\boldsymbol{A}$,$\boldsymbol{B} \in \mathbb{R}^{n \times n}$ are state matrices [14]. The roots of the state space equation are the solutions in s to

$$s\boldsymbol{I} = \boldsymbol{A}e^{-s\tau} + \boldsymbol{B}.$$

Expanding by $\tau e^{(s\boldsymbol{I}+\boldsymbol{B})\tau}$, this transcendental characteristic equation can be rewritten as

$$(s\boldsymbol{I} - \boldsymbol{B})\, \tau e^{(s\boldsymbol{I}+\boldsymbol{B})\tau} = \boldsymbol{A}\tau e^{\boldsymbol{B}\tau}.$$

The equation has now the form to apply the Lambert W function. Obviously, $W\left(\boldsymbol{A}\tau e^{\boldsymbol{B}\tau}\right) = (s\boldsymbol{I} - \boldsymbol{B})\,\tau$.

Lemma 4 ([14]). The roots of $\dot{x} = \boldsymbol{A}\, x(t - \tau) + \boldsymbol{B}\, x$ are the elements of $\sigma\left(\frac{1}{\tau} W\left(\boldsymbol{A}\tau e^{\boldsymbol{B}\tau}\right) + \boldsymbol{B}\right)$.

II. The Smith Synchronizer

The control introduced in this paper uses a traditional Smith Predictor [23], [24] with the output signal of a model of the plant as reference. This system could also be considered as a signal generator. The control is depicted in Fig. 1.

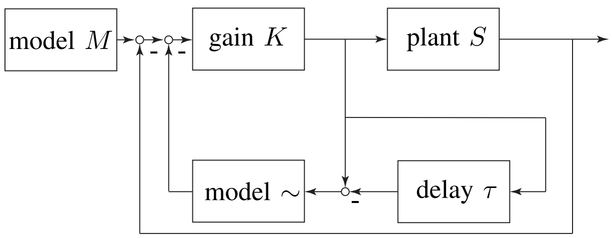

Fig. 1. Smith Synchronizer

Definition 8. The control topology in Fig. 1 is henceforth called Smith Synchronizer.

Furtheron, the reference model will be indexed with M (master), the plant with S (slave), and the auxiliary model with $\sim$. The gain is a scalar K. Let

$$\dot{x}_i = f(x_i) + \boldsymbol{B}\, u_i$$

$$y_i = \boldsymbol{C}\, x_i$$

be the state space equation of x_i, $i = M, S, \sim$, where $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ is the input matrix and $\boldsymbol{C} \in \mathbb{R}^{n\times n}$ the output matrix. It should be noted that these matrices can be expanded by zero rows or columns when they have lower dimensions.

A. Modal Analysis

Formalizing Fig. 1, the input vectors can be rewritten as a linear combination of all state vectors, as stated in the introduction. In doing so, all inputs of S will be considered as delayed inputs, as it represents the delayed plant, that is supposed to synchronize with M, so that

$$u_M = 0,$$

$$u_S = \boldsymbol{C}K\left[x_M(t-\tau) - x_\sim(t-\tau) - x_S(t-\tau)\right],$$

$$u_\sim = \boldsymbol{C}K\big[x_M - x_\sim - x_S -$$

$$-x_M(t-\tau) + x_\sim(t-\tau) + x_S(t-\tau)\big].$$

Using these relations and substituting them into the state space equations of the respective systems, the system dynamics are of the form

$$\dot{x}_M = f(x_M)$$

$$\dot{x}_S = f(x_S) + \boldsymbol{BC}K\left[x_M(t-\tau) - x_\sim(t-\tau) - x_S(t-\tau)\right]$$

$$\dot{x}_\sim = f(x_\sim) + \boldsymbol{B}\,\boldsymbol{C}K\big[x_M - x_\sim - x_S -$$

$$-x_M(t-\tau) + x_\sim(t-\tau) + x_S(t-\tau)\big].$$

Now let $\chi = [x_S\ x_M\ x_\sim]^{\mathrm{T}} \in \mathbb{R}^{3n}$ be the all-state vector of the Smith Synchronizer and $\dot{\chi} = \frac{d\chi}{dt}$ its time-derivative. Then the dynamics of χ have the form

$$\dot{\chi} = \begin{bmatrix} f(x_S) \\ f(x_M) \\ f(x_\sim) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -K & K & -K \end{bmatrix} \otimes \boldsymbol{B}\,\boldsymbol{C}\,\chi +$$

$$+ \begin{bmatrix} -K & K & -K \\ 0 & 0 & 0 \\ K & -K & K \end{bmatrix} \otimes \boldsymbol{B}\,\boldsymbol{C}\,\chi(t-\tau).$$

The coupling configuration matrices will be substituted by $\boldsymbol{\Gamma}_t$ and $\boldsymbol{\Gamma}_{t-\tau}$, respectively.

The crucial property of the coupling that is induced by the given control is that it is nonlinear. In existing literature, linear matrix inequalities are used to show the synchronizability of such couplings [5]–[12]. This paper will conduct a modal analysis of the coupling dynamics $\dot{\psi} = \boldsymbol{\Gamma}_t\psi + \boldsymbol{\Gamma}_{t-\tau}\psi(t-\tau)$, where $\psi \in \mathbb{R}^3$ is a variable describing the solutions of the three systems relative to each other, just as coupling configuration matrices do. Hence, if ψ has a synchronized state, then so has the Smith Synchronizer.

Lemma 5. The Smith Synchronizer induces a synchronized state for M and S.

Proof: Using Lemma 4, the eigenvalues of ψ are the elements of the set $\sigma\left(\frac{1}{\tau}W\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) + \boldsymbol{\Gamma}_t\right)$. Let $\boldsymbol{\Lambda} = \mathrm{diag}\left\{\sigma\left(\frac{1}{\tau}W\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) + \boldsymbol{\Gamma}_t\right)\right\}$, then

$$\boldsymbol{V}^{-1}\left(\frac{1}{\tau}W\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) + \boldsymbol{\Gamma}_t\right)\boldsymbol{V} = \boldsymbol{\Lambda}$$

denotes the well known similarity transform, where $\boldsymbol{V}$ is the modal matrix of $\frac{1}{\tau}W\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) + \boldsymbol{\Gamma}_t$ with columns being the modes of ψ. Applying Sylvester's formula,

$$e^{\boldsymbol{\Gamma}_t\tau} = \boldsymbol{Q}e^{\boldsymbol{D}}\boldsymbol{Q}^{-1},$$

where $\boldsymbol{D} = \mathrm{diag}\left(\sigma\left(\boldsymbol{\Gamma}_t\tau\right)\right)$ and $\boldsymbol{Q}$ the modal matrix of $\boldsymbol{\Gamma}_t\tau$. Now, because $\sigma\left(\boldsymbol{\Gamma}_t\tau\right) = \{0, 0, -\tau K\}$, its eigenvectors build the matrix

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

so that the matrix exponential results in

$$e^{\boldsymbol{\Gamma}_t\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{-\tau K} - 1 & -e^{-\tau K} + 1 & e^{-\tau K} \end{bmatrix}.$$

The argument of the Lambert W function is the matrix

$$\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau} = \begin{bmatrix} -\tau K e^{-\tau K} & \tau K e^{-\tau K} & -\tau K e^{-\tau K} \\ 0 & 0 & 0 \\ \tau K e^{-\tau K} & -\tau K e^{-\tau K} & \tau K e^{-\tau K} \end{bmatrix}.$$

Obviously, $\mathrm{rank}\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) = 1$ and $\sigma\left(\boldsymbol{\Gamma}_{t-\tau}\tau e^{\boldsymbol{\Gamma}_t\tau}\right) = \{0, 0, 0\}$. The eigenvalue zero has the algebraic multiplicity three but only a geometric multiplicity of two. There is no similarity transform to this matrix, so that is is not possible to apply Sylvester's formula. But, since the matrix is nilpotent with $\left(\mathbf{\Gamma}_{t-\tau}\tau e^{\mathbf{\Gamma}_t\tau}\right)^{\varphi} = \mathbf{0}\ \forall\ \varphi > 1$, the matrix Lambert W function can instead be expressed as a fast converging sum. Since the principal branch of the Lambert W function is defined as

$$W_0\left(z\right) = \sum_{n=1}^{\infty} \frac{\left(-n\right)^{n-1}}{n!} z,$$

it converges after the first element in this case, so that

$$W\left(\mathbf{\Gamma}_{t-\tau}\tau e^{\mathbf{\Gamma}_t\tau}\right) = \mathbf{\Gamma}_{t-\tau}\tau e^{\mathbf{\Gamma}_t\tau}.$$

The matrix that defines the latent roots and modes of the Smith Synchronizer has therefore the form

$$\frac{1}{\tau}W\left(\mathbf{\Gamma}_{t-\tau}\tau e^{\mathbf{\Gamma}_t\tau}\right) + \mathbf{\Gamma}_t =$$

$$= \begin{bmatrix} -K\mathrm{e}^{-\tau K} & K\mathrm{e}^{-\tau K} & -K\mathrm{e}^{-\tau K} \\ 0 & 0 & 0 \\ K\left(\mathrm{e}^{-\tau K}-1\right) & -K\left(\mathrm{e}^{-\tau K}-1\right) & K\left(\mathrm{e}^{-\tau K}-1\right) \end{bmatrix},$$

with $\sigma\left(\frac{1}{\tau}W\left(\mathbf{\Gamma}_{t-\tau}\tau e^{\mathbf{\Gamma}_t\tau}\right) + \mathbf{\Gamma}_t\right) = \{-K, 0, 0\}$ and the respective modal matrix

$$\boldsymbol{V} = \begin{bmatrix} -\frac{e^{-\tau K}}{e^{-\tau K}-1} & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The mode corresponding to the third eigenvector is a synchronized state of S and M, that is undamped. All other modes have eigenvalues with non-positive real parts. However, these modes do not affect the synchronized movement, since they all have a zero entry in the second row. The $k \neq 0$ branches of the Lambert W function can be ignored, since the real parts of the roots are strictly monotonically decreasing with increasing k. $\square$

B. Evaluation

The Smith Synchronizer is said to have a synchronized state, but also has an asynchronous mode corresponding to an undamped latent root of the variation. The asynchronous mode is proportional to the time solution of $\sim$. From Fig. 1, the input of this system is porportional to the difference of the delayed and the non-delayed solution in x. The structure would therefore induce synchronizability for $\|x\left(t-\tau\right) - x\left(t\right)\| = 0$.

Lemma 6. The Smith Synchronizer is not synchronizable for all $\tau > 0$.

Proof: The second mode of the Smith Synchronizer is undamped with the eigenvector $[1\ 0\ -1]$. Without that mode, systems S and M are synchronizable. Now there is an additional motion of S antiproportional to the motion of $\sim$, so that $\|x_S - x_M\| \propto x_\sim$. The input signal of $\sim$ is proportional to $1 - e^{-\tau s}$, that is $> 0\ \forall\ \tau > 0$. $\square$

Definition 9. A system with a synchronized state, nonpositive latent roots and at least one more latent root with real part zero is said to achieve ragged synchronization. The remaining error $\|x_i - x_j\| = e$ is called synchronization error.

Lemma 7. The Smith Synchronizer is said to achieve ragged synchronization with a synchronization error bound by $\|x\left(t-\tau\right) - x\left(t\right)\|$.

The mathematical evaluation of the synchronizability yielded results that can be interpreted physically. Whenever the dynamics of the system are faster then the delay, the Smith Synchronizer is not suitable for monitoring M instead of S. It is also possible to summarize this capability in a numerical value ϑ, that will be normalized to

$$\vartheta = \frac{\|x\left(t-\tau\right) - x\left(t\right)\|}{\|x\left(t\right)\|}.$$

It can be noted, that this value is quite similar to the transparency coefficient known from hardware-in-the-loop publications [25], [26]. When ϑ is small, that is for very slow functions, the Smith Synchronizer may lead to synchronous behaviour. Those slow functions are often highly stable functions.

As an example, the well known test function $\dot{x} = -x + u$ will be applied to the Smith Synchronizer. Though it might not me a nonlinear function, it is still a good example for employing the Smith Synchronizer, as its synchronous state becomes more apparent to the reader. The solutions of the states of the three systems are depicted in Fig. 2. In the underlying simulation, arbitrary initial values $x_M\left(0\right) = -1$, $x_S\left(0\right) = 1$, $x_\sim\left(0\right) = 0$ have been applied to all three systems. After $t = \tau$, the states of M and S converge to

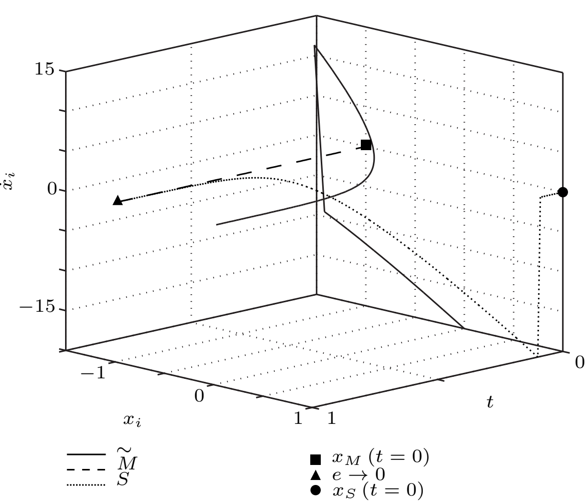

Fig. 2. Converging Modal Attractors

$e \to 0$ due to the behaviour of $\sim$. It now results from the modal analysis that for the deduced conditions, the damping of the first asynchronous mode is proportional to K. The test function from Fig. 2 can therefore also be evaluated for different K. The solution for e versus time is plotted in Fig. 3. According to the time solution of the test function, the time solution of e has an exponential behaviour with an exponent antiproportional to K.

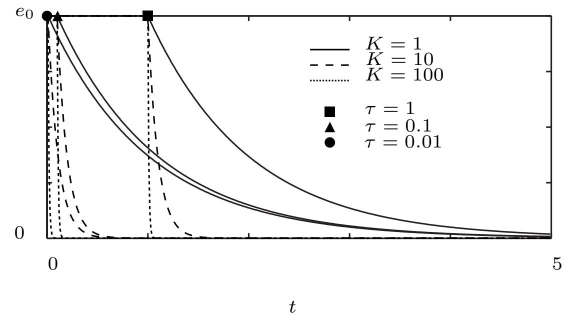

Fig. 3. Synchronization Errors

This example shows that the mathematical framework used in synchronization theory can be applied to the coupling of systems that is put into practice in control theory. It is also possible to analyze that systems with relatively slow dynamics induce a transparency coefficient that is small enough to achieve ragged synchronization with a Smith Predictor.

III. CONCLUSIONS

A framework for the modal analysis of controls has been established and criteria for a synchronized state and synchronizability have been cited. The Smith Synchronizer was defined and its synchronized state proven, before its modes have been analyzed and evaluated according to its suitability for different systems. The physical meaning of the mathematical results was interpreted and illustrated by the time solutions of the test function $\dot{x} = -x + u$. Further exploitations could expand the Smith Synchronizer to a control topology where the reference M is given by a signal generator, that is not governed by the same equation as S and $\sim$. This expansion would make the underlying theory applicable to a wider range of problems.

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