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ON SIZE AND CONTROL: THE USE OF DIMENSIONAL ANALYSIS IN CONTROLLER DESIGN

BY

SEAN N. BRENNAN

B.S., New Mexico State University, 1997
B.S., New Mexico State University, 1997
M.S., University of Illinois, 1999

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SEAN N. BRENNAN

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Abstract

The central tenet of this thesis is that the tools of dimensional analysis offer significant benefits to control-problem formulation and controller design. This argument is developed using the task of vehicle lateral positioning control at highway speeds as a motivating example. The vehicle control problem is framed in a dimensionless form, solved using robust control theory, and confirmed experimentally on a vehicle. It is shown that control techniques usually used for dimensional systems are also valid for a dimensionless representation, but that the dimensionless approach allows direct extension of the control results to other vehicles of arbitrary make or size. Remarks discuss fundamental limitations with vehicle control as well as extensions of dimensional analysis to other systems.

The dimensionless approach not generally used in the control field was motivated by the use of the Illinois Roadway Simulator to test vehicle control techniques on 1/8- to 1/14-scale vehicles driving on a large treadmill. The vehicle/treadmill system is analogous to an aircraft/wind-tunnel system. A key concern in this approach was the validity of the experimental model dynamics, and hence the resulting controller, with regard to dimensional scaling. Application of the well-known Pi-Theorem to the vehicle model yielded a set of dimensionless parameters, and calculation of these parameters for over 700 vehicles revealed well-defined distributions. In addition to confirming dynamic similarity of the scale vehicle, these distributions provide a numerical measure of average and range in vehicle behavior for all vehicles. This allowed a precise, numeric definition of an 'average' vehicle, as well as a tight numerical range in vehicle dynamics. Extension of the parameter-based-approach to differential-equation models allowed the development of a robust vehicle controller that is portable from vehicle-to-vehicle.

More importantly, dimensional analysis has yielded significant and novel insights into standard problems of control theory. Specifically, dimensional scaling constraints are shown to correlate to invariant Bode sensitivity conditions implicit in the governing model. Similar invariant equations have been studied under the framework of electrical network analysis since the 1960's; this thesis provides perhaps the first generalized solution. Extensions of the sensitivity result to problems of gain-scheduling, model reduction, adaptive control and identification, and nonlinear systems analysis are presented in both a theoretic context and in the solution of many different examples.

To My New Family

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Chapter 1

Introduction

> …If we tried building ships, palaces, or temples of enormous size, yards, beams and bolts would cease to hold together; nor can Nature grow a tree nor construct an animal beyond a certain size, while retaining the proportions and employing the materials which suffice in the case of a smaller structure. The thing would fall to pieces of its own weight unless we either change its relative proportions, which will at length cause it to become clumsy, monstrous and inefficient, or else we must find new material, harder and stronger than what was used before. Both processes are familiar to us in nature and in art, and practical applications, undreamed of by Galileo, meet us at every turn in this modern age of cement and steel.
>
> - Wentworth Thompson, *On Growth and Form*, 1917, reviewing Galileo's work on scaling theory (McMahon and Bonner, 1983)

This thesis deals with the synthesis of two topics: dimensional analysis and control theory. To make a crude distinction, the first field specifies how to obtain measurements; the second addresses how to use measurements to influence a system. More exactly, dimensional analysis studies functional relationships and methods to simplify their representation by specifically analyzing dimensional and parameter scaling. Control theory studies methods to manipulate system behavior by modifying inputs to the system based on measurements taken from the system.

Obviously, there is a strong coupling between control theory and dimensional analysis, yet there is oddly little overlap in approach between the two studies. While there are thousands, if not tens of thousands, of researchers actively studying control theory, there are very few who study dimensional analysis. There are even fewer who study this topic in the context of system control. Within the past decade, outside of the work by the author and colleagues, there are only a handful of citations within the area of control that utilize dimensional analysis. In each of these cases, such analysis was intended to demonstrate system equivalence between a scale model and a full-sized prototype (Ghanekar, Wang, and Heppler, 1997) (Liu and Lee, 1998), and little consideration has been given to extending the analysis into new control fields. The unified approach presented in this thesis was quite serendipitous and the result of a very unique control task discussed in Chapter 2.

As the usage of computer-based simulations increasingly dominates over testing physical hardware, the understanding and usage of scale physical models is a dying art, and the requisite mathematical basis of comparing systems of different sizes is no longer taught to an extent that would be useful to a control theoretician. Yet, analysis of very complex simulations of physical systems is quickly becoming a deep and central problem in implementing many new control strategies. Dimensional analysis was in the near past the primary, pre-computer, pre-simulation method of analyzing complex systems that were unsuitable for direct experimentation. This centuries-old dimensional approach provides analytic nuances that are just as relevant to silicon-based simulations as they are to scale-sized physical experiments. The nuclear reactors and space rockets in use today were originally designed with scale models using dimensional techniques. It is a central hypothesis of this thesis that the present and future automation of such systems can be greatly improved and simplified by understanding *why* dimensional techniques were so useful in the past.

To understand what to expect from a dimensional approach, we must generalize results observed from previous usage. First, the past has taught us that a clear concept of dimensional similitude is a general prerequisite to developing insightful and useful scientific and engineering theory. A litany of the participants in the field of dimensional analysis is also a litany of the world's best scientists and engineers, and it is unlikely that this is a coincidence. Second, dimensional analysis has demonstrated its utility in studying phenomena that that are not yet completely understood, but whose general principles are understood. This last point typifies the present controls environment, when perhaps more than at any other time in human history an understanding of dimensional scaling seems crucial. With today's technology, mankind is able to manipulate and control systems ranging on scales from the atom to the global environment, with complexities ranging from millions, to billions, perhaps trillions of interconnected subsystems. While the fundamentals of control theory have been long established, the generalization of fundamental principles to problems of various scales becomes daunting without a system-generalizable control approach.

If one extends the present growth of computer-based physical systems into the near future, one finds that the number of processes that can be automated rapidly grows past the human ability to design and tune controllers. In such a future, expert intervention in the control loop analysis will become the limiting factor of increasing automation. Any method that can generalize the analysis of control loops for common physical systems in a mathematical context will certainly be beneficial.

Because so few in the field of control theory are aware of dimensional analysis, a large portion of this thesis is focused of development of this topic in a context of control theory. It is assumed that the reader is familiar with general systems analysis and the basic concepts of control. Because this thesis is not a textbook, some topics may be presented with little or no supporting development, while others are presented in detail. The author sends his apologies to the reader if some background is missing, but it is hoped that the number and extent of examples in this thesis may serve to illustrate the primary concepts where a full analytical development may be incomplete.

1.1 A Simple Example

Perhaps the most general example system used in systems analysis is the mass-spring-damper, shown in the diagram below. This example was chosen because so many systems of so many size scales are said to be mass-spring-dampers, although no system in truth can act like the ideal model usually considered for this description. From the oscillation of buildings, the motion of vehicle suspension systems, the bounce of an object on a bed, the dynamics of a hydraulic valve, or the stability of planetary motions, the analogy to the mass-spring-damper is often made.

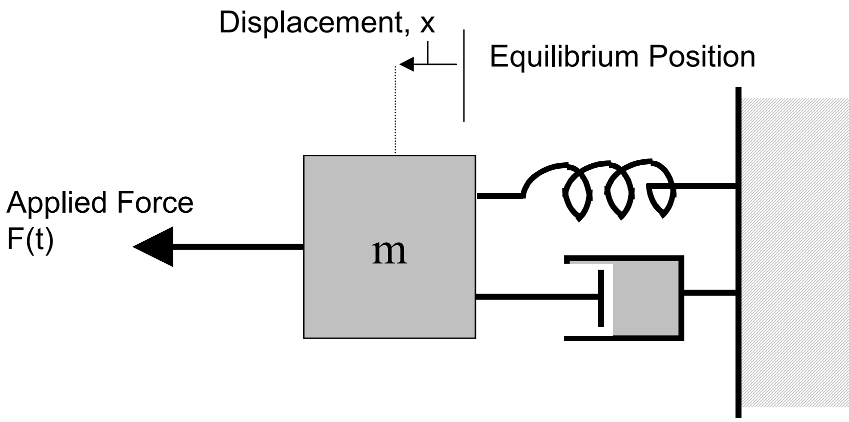


Figure 1.1: Mass-spring-damper

The differential equation governing the theoretical motion of the mass of an ideal mass-spring-damper is given by Equation 1.1:

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+k\cdot x=F(t) \tag{1.1}$$

As this thesis is concerned with physical dimensions, we first note that all the terms in the equation have dimensions of force. Clearly, based on the range of mass-spring-damper examples in the literature, an exact similarity of force, or motion, or mass, or of any parameter is not required. We therefore question how one might know that two systems might share a common description of "mass-spring-damper". The question becomes increasingly significant when we note that the representation of a single system can yield a very large range of mathematical equations.

For illustration, let us assume that we are using the *Meters-Kilograms-Seconds* (MKS) unit system, that the sole governing equation is Newton's Law (this assumption is implicit in the form of the equation), and that each coefficient in the equation has a unit measure in the MKS system.

$$m=1\ kg, \beta=1\ \frac{kg}{sec}, k=1\ \frac{kg}{sec^2} \tag{1.2}$$

If we use the parameter of Equation 1.2, the poles become s = -0.5000 ± 0.8660. However, the poles may seem to change arbitrarily when measured in different units:

	m	β	k	$poles$
$meters-kilograms-sec.$	1	1	1	-0.5000 ± 0.8660
$miles$ - $kilograms$ - $hours$	1	3600	$1.296E7$	-1800 ± 3117.7i
$meters-ounces-millisec.$	35.27	$3.527E-2$	$3.527E-5$	-0.005 ± 0.008660i
$meters-tons-hours$	$1.102E-3$	3.9683	14285	-1800 ± 3117.7
⋮	⋮	⋮	⋮	⋮

(1.3)

Given the four different unit systems above, and consequently the four different transfer function representations, how would one know that these four numerically different systems were identical?

Additionally, there exist cases where the system measurements are not identical, but the mathematical analysis is clearly similar. For instance, if the parameters of a second system are:

$$m = 1\ g, \beta = 1\ \frac{g}{msec}, k = 1\ \frac{g}{msec^2} \tag{1.4}$$

Based on the coefficients, the second system is at a minimum mathematically identical to the first system if the units are ignored. Unfortunately, if one represents system 1.4 in a unit system of *meters-kilograms-seconds*, then it would again be unclear whether the two systems share a common model similarity. This topic is obviously very relevant toward developing a unified control analysis for the two systems.

This simple example raises interesting issues for a control theorist. Clearly, the transfer function representation of a physical system is NOT unique because simple dimensional transformations can change this representation significantly. Additionally, two different systems may share a common control design approach, yet finding compatible dimensions between the two systems to illuminate this fact may not be obvious. Finally, it remains unclear whether there might be a *preferred* unit system for controller design that might simultaneously address the above issues. Especially with regard to the last point, this thesis explores how the choice of a dimensioning system may be best selected from a controller-design standpoint.

1.2 Statement of Proposed Research

The goal of the current research is to specifically examine what improvements can be obtained in the representation and control of dimensional systems by incorporating dimensional analysis into system analysis and controller design. A central feature of this work is to obtain system representations that are valid, independent of the choice of unit systems. Such independence imposes constraints on the governing equation form, and consideration of these dimensional constraints (i.e. 'Dimensional Analysis') is a central topic of this thesis. The resulting treatment is intended to develop methods capable of describing and developing control laws useful for an entire system 'class', independent of physical size or specific plant within the class.

To further illustrate the notion of a system class, we note that with the previous example that the notion of a mass-spring-damper encompasses a generalizable but well-defined class of systems. A controller developed for one mass-spring-damper should be able to readily generalize to all other mass-spring-damper systems under certain constraints. Methods to generalize results in a system-independent manner are considered in this study. This approach unifies many different control topics whose interdependence may be unclear in the traditional controller approach.

As a specific and motivating example, the field of vehicle chassis control is presented throughout this work. Specifically, various methods of dimensional analysis are used to improve the system design, uncertainty description, and closed-loop control of the 'class' of all passenger vehicles. The emphasis in this analysis is on a strong experimental approach, and such emphasis has enlightened significant possible improvements in control theory. Equivalently, we have discovered what are felt to be completely new topics in the field of dimensional analysis. Such topics include the equivalency of sensitivity functions as guaranteed by physical similarity, stability of dimensional systems, and the generation of system perturbations as defined by the span of system classes. Additional examples are provided throughout this thesis based on well-known control and dimensional problems such as the mass-spring-damper example just presented.

1.3 Objectives and Significance

The specific goals of this research are as follows:

- Represent systems such that plant parameters do not have dimensions. The dimensionless groupings of the plant parameters will be referred to as Pi parameters, and are obtained by careful grouping of traditional, dimensioned parameters,
- Show that Pi-parameter equivalence between systems guarantees dynamic similitude, while traditional system analysis properties such as pole locations do not,
- Define dynamic similitude in a sensitivity statement generalizable to all control problems,
- Show that dimensional analysis can reveal duality between many different types of control problems that is otherwise hidden in traditional differential-equation representations,
- Discuss appropriate selection of the Pi parameters to maximize the portability of a controller design,
- Discuss why dimensional scaling problems rarely arise because of 'accidental' methods of dimensional analysis that have been implicitly used in the field,
- Obtain distributions of the Pi parameters. These distributions:
    - Provide insight into the design constraints of the system, whether the constraints are intentional or otherwise,
    - Simplify analysis of open-loop stability by accounting for parameter interdependence,
    - Describe the average member of the distribution, and thus indicate how to select and design an experimental system to be the most 'average' representation of a dynamic class,
    - Describe the range of parameter variation that is to be expected from a dynamic class, allowing tight and non-conservative uncertainty representation,
    - Allow comparisons between plants at different research institutions, thus allowing identification of anomalous systems before wasting significant identification and controller design effort,
- Design controllers robust to inter-class parameter variation, thus producing a single control law suitable for an entire dynamic class,

- Demonstrate all of the above goals using vehicle chassis control as an example system, including experimental implementation results,
- Examine nonlinear systems using dimensional analysis to examine feasibility and possible alternative dimensional representations more suited for control.

The significance of the above goals should be obvious since every physical plant is necessarily a dimensional system. Any control approach on a physical system could therefore potentially benefit from the tools of dimensional analysis.

1.4 Outline of Remaining Chapters

The second chapter presents an introduction to the motivating problem of this thesis: the design of control systems to benefit vehicle behavior during highway driving. This problem was studied using a scale vehicle system on a treadmill, similar to how a wind-tunnel is used for studying scale aircraft behavior. An inquiry into the dimensional scaling aspects of control theory related to this problem was natural, and this thesis is the result of this investigation.

The third chapter introduces the mathematics of dimensioned quantities. There are two goals of this chapter: first, to present a very basic yet intuitive tutorial into dimensional analysis and second, to introduce notation used throughout the remainder of this thesis. Readers already familiar with dimensional analysis may skip the first portions of this chapter, as only the last three sections are used within the remainder of the thesis.

The fourth chapter discusses sensitivity analysis and how dimensional analysis plays a central role in this critical topic of control theory. The primary points of this chapter are based on the observation that the Bode sensitivity functions of dimensioned systems are highly coupled. This fact was well-known within the control community regarding network sensitivities. However, the fact that this result relies primarily on dimensional analysis and can be extended to other and arbitrary physical systems is a true contribution of this thesis. This chapter serves to develop this topic further in a theoretical context to generalize what improvements to control theory might be gained.

The fifth chapter investigates specific areas of linear control theory that rely on parameter-dependent methodologies, and would therefore specifically benefit from dimensionless parametric representations of dynamic systems and control loops. This chapter presents the topic largely in an example-oriented approach, focusing on the topics of stability analysis, model reduction, gain scheduling, model identification, and robust control.

The sixth chapter extends the analysis of linear systems to problems whose control laws are non-parametric. Here, the advantages of dimensional analysis are developed in a frequency-domain approach. The illustrating example of this chapter is the solution of a vehicle-control problem whose model uncertainty is specified via frequency bounds on the model.

The seventh chapter summarizes the primary results and conclusions of this thesis. Future work is introduced, specifically the use of dimensional analysis on nonlinear systems. Natural extensions of results from linear systems are discussed, including localized, static nonlinearities that are coupled with LTI (linear time-invariant) systems such as rate-limited linear systems. A dimensional analysis based discussion of the feedback-linearization method of the control of a SCARA robot is presented. Preliminary results are presented for these implementations. Additional discussion is given to other potential research areas in the area of control theory as related to dimensional analysis.

1.5 References

1. Ghanekar, Milind, David W. L Wang, and Glenn R. Heppler. "Scaling Laws for Linear Controllers of Flexible Link Manipulators Chacterized by Nondimensional Groups." IEEE Transactions on Robotics and Automation 13.1 (1997): 117-27.
2. Liu, T. S., and J. Y. Lee. "A Similarity Method for Minature Mobile Robots." ASME Journal of Dynamic Systems, Measurement, and Control 120.March (1998): 15-21.
3. McMahon, Thomas A., and John Tyler Bonner. On Size and Life. New York: Scientific American Books, Inc., 1983.

Chapter 2
Motivation: The Illinois Roadway Simulator and the Scalability of Vehicle Dynamics

The study of dimensional analysis began with the author's work on the Illinois Roadway Simulator, which utilizes reduced-scale vehicles running on a treadmill to emulate full-sized vehicles on a highway. Current practice in the vehicle dynamics and control community is to validate detailed simulation results using a full-sized vehicle. For university-based research, this approach is often prohibitively expensive, as well as dangerous, especially when considering testing the linearity of a model via aggressive driving situations for which simulation matching may be poor. Unfortunately, it is precisely under these aggressive driving conditions that the controller will be most expected to perform correctly. To circumvent the cost and inherent danger in testing aggressive vehicle controllers using full-sized vehicles, a scale vehicle testbed was developed for use as an evaluation tool to bridge the design gap between simulation studies and full sized hardware: the Illinois Roadway Simulator (IRS) (Brennan, 1999).

2.1 Motivation for a Scale Testbed

To motivate the use of a scale testbed, one need only examine the rich history of dynamic studies of moving vehicles. The Froude number was so named in recognition of a ship construction engineer who developed a methodology for evaluating ship designs. A catastrophic design failure occurred causing a cargo ship to be decommissioned immediately after construction because its operating costs were more than double the costs predicted by conventional theory. Using scale ships to conduct simulations, Froude soon described a previously unmodeled but significant relationship between wave production and energy losses.

The success of Orville and Wilbur Wright in developing the first heavier-than-air aircraft can be attributed primarily to their methodical testing of scale aircraft in the first-ever wind tunnel of their own construction (McMahon and Bonner, 1983). The lunar rover designs of the 1960s were tested using scale vehicles driven across uneven surfaces; photographs of these simple vehicle designs reveal a direct lineage to the Mars Sojourner robot (Bekker, 1969). Clearly, the use of scale systems has been beneficial to the measurement of open-loop dynamics and design of vehicle systems; however, use of this same approach for closed-loop vehicle controller evaluation has been lacking.

Recent investigations using scaled vehicles for control validation (Sampei, *et al.*, 1995; Matsumoto and Tomizuka, 1992) have mostly involved moving the vehicles along some fixed surface, which naturally raises a host of interfacing and sensing issues. The IRS, by comparison, is an experimental testbed consisting of scaled vehicles running on a simulated road surface, where the vehicles are held fixed with respect to inertial space and the road surface moves relative to the vehicle. An analogy would be wind tunnel testing of aerospace systems.

The IRS provides several advantages over full-scale vehicle testing. First, the availability of scale components makes construction faster and cheaper; a new vehicle/test design of moderate complexity can be built from scratch in about 100 person-hours for less than $2000. The durability of these vehicles and the ability to intervene during an accident makes testing safe and repeatable. The scheduling and use of public or private roadways is not required, and no drivers or pedestrians are put at risk during testing of aggressive vehicle controllers. The simulated roadway surface can be varied quickly and easily to emulate changing road surfaces, or kept uniformly even for as long as testing requires. The IRS offers considerable sensing and actuation flexibility and a choice between on- or off-vehicle mounting of sensors. The vehicle parameters can be varied while the vehicle is running. The vehicle can intentionally be crashed, spun, nudged, lifted, and/or otherwise destabilized in a repeatable and easy manner. Finally, testing with the IRS has shown much dynamic similitude between scale and full-size vehicles.

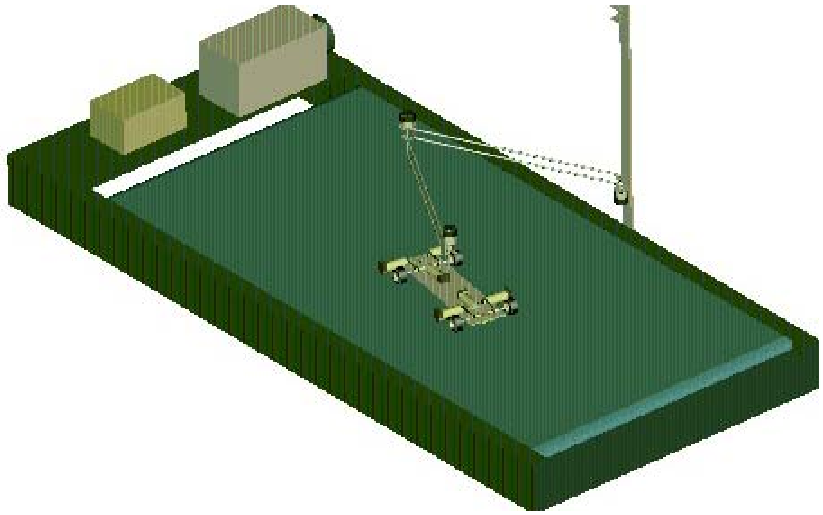


Figure 2.1: CAD diagram of the Illinois Roadway Simulator

The IRS's scaled roadway surface consists of a 4 x 8 ft treadmill capable of top speeds of 15 mph. Scale vehicles are run on the treadmill via multiple wall-mounted transmitter systems operating between 50 and 100 MHz or via direct tether connections to the vehicle. The remainder of the IRS consists of a driver console, DSP- and PC-based interface computers, A/D and D/A converters, a significant amount of electronic interface equipment, a vehicle position sensor system, and the vehicles. The vehicle controller hardware loop can use driving signals available via a manual driver console or from a computer-generated signal. All external signals are sampled or updated at 1 kHz via Analog Devices PCI RTI-815 analog I/O boards, Analog Devices RTI 802 analog output boards, and U.S. Digital encoder boards. The driving voltage and steering voltage signals are converted into a current signal proportional to input voltage via Servo Systems linear amplifiers and sent in pulse-width-modulated form to brush-commutated DC motors that act as steering and drive motors. Because the motor torque is proportional to input current for DC motors, the wheel and steering torque can be commanded directly. Although this direct method torque input is not currently practiced on conventional vehicles, future vehicle designs using hybrid electric/internal-combustion engine drive systems will likely have this capability. Further, the drive motors on IRS scale vehicles can be made to emulate a conventional drive/braking system by modifying the control algorithm to exhibit powertrain and braking dynamics.

The treadmill road surface velocity is varied to maintain the vehicle position with respect to an inertial reference point. The roadway speed is monitored via an optical encoder. To maintain the vehicle on the treadmill, the vehicle's inertial position is used as feedback for a controller that outputs a voltage signal to the treadmill. The treadmill uses an industrial motor controller that converts the input voltage level to a reference speed and adjusts the DC drive motor current to match this speed accordingly. Figure 2.1 gives a representation of the entire system. Information regarding the vehicle position sensing and longitudinal positioning dynamics can be found in the author's M.S. thesis (Brennan, 1999).

All data-acquisition (DAC) and control features are handled via Wincon®, a windows-based control program that runs real-time code generated by MATLAB/Simulink's Real-Time Workshop® toolbox. Custom drivers were written at the University of Illinois at Urbana-Champaign to communicate with the Analog Devices boards. This Wincon interface eliminated lower level C-programming and allowed all functions to be handled with a graphical user interface (GUI) similar to Simulink. Additionally, it provided for real-time viewing of data.

2.2 A Literature Survey of Vehicle Control

The scope of the vehicle control literature is quite extensive, and the interested reader is referred to review articles (Tomizuka and Hedrick, 1995; Furukawa, Moshimi, and Masato, 1997) for appropriate summaries of the field. In general, two types of vehicle chassis control are employed: control of the lateral position of the vehicle, and control of the yaw rate. The intent of lateral position control is for a control algorithm to maintain the vehicle's position on the road. However, lateral positioning algorithms in general do not account for driver input during the driving task. Hence, such studies require specially equipped vehicles and/or road surfaces. A significant amount of work has been done and is continuing in developing such Automated Highway Systems, yet commercial use of such research remains to be seen.

Increased vehicle performance, robustness, and stability motivate the commercial use and experimental study of yaw rate feedback. This is summarized best by Hatipoglu et al.: "Most of the vehicle disturbances, non-smooth actuator nonlinearities, unmodeled imperfections stemming from vehicle-road interactions and uncertainties regarding system parameters can be by-passed through the yaw rate measurement" (Hatipoglu, Redmill, and Ozguner, 1926). Most of the recent advances in commercial stability and performance have been in the use of yaw-rate feedback and in the incorporation of such feedback with subsystems such as ABS and Traction Control. Other authors have appropriately dealt with the system nonlinearities using direct yaw control, most notably in (Shibahata, Shimada, and Tomari, 1993; Pilutti, Ulsoy, and Hrovat, 1995; Nagai, Hirano, and Yamanaka, 1997).

The work by the author on vehicle chassis control has been extensive, covering the areas of hardware and vehicle system identification (Brennan, 1999; Brennan and Alleyne, 1998), Hardware-In-The-Loop simulations (Brennan and Alleyne, 1998), Driver-Assist yaw-rate control using rear-wheel steering (Brennan and Alleyne, 1999; Brennan, 1999), Driver-Assist yaw-rate control using torque-steering (Brennan, 1999), Model Reference control (Brennan and Alleyne, 1999; Brennan, 1999; Brennan and Alleyne, 1998), and Robust lateral positioning control (Brennan and Alleyne, 2002; Brennan and Alleyne, 2001; Brennan and Alleyne, 2000). Because this work is extensive, only selected applicable results are presented in this discussion.

2.3 System Dynamics Description

Modeling of the vehicle dynamics is accomplished by fixing a coordinate system to the center of gravity (CG) of the vehicle and solving for the equations of motion. Roll, pitch, bounce, aerodynamics, and deceleration dynamics are neglected to simplify the vehicle dynamics to two degrees of freedom: the lateral position and yaw angle. The model is further simplified by assuming that each axle shares the same steering angles, and that consequently each wheel produces the same wheel angle steering forces. The resulting dynamic model is known as the bicycle model, because the dynamics conceptually model a bicycle whose motion is constrained to in-plane motion.

Traditionally, the bicycle model is formulated in transfer-function form using the front wheels as steering inputs. The use of this model is explained in detail by Dugoff, Fancher, and Segel (Dugoff, Fancher, and Segel, 1970). Although the bicycle model is relatively simple, a research project conducted in partnership between Ford and the University of Michigan Transportation Research Institute (UMTRI) verified that the bicycle model remains a good approximation for full-size vehicle dynamics as long as accelerations are limited to 0.3 g's (LeBlanc, *et al.*, 1996). In the implementation of a controller based on the linear model, care must be taken to incorporate a model-switching method when tire forces saturate, as measured by the ABS or similar wheel slip sensor.

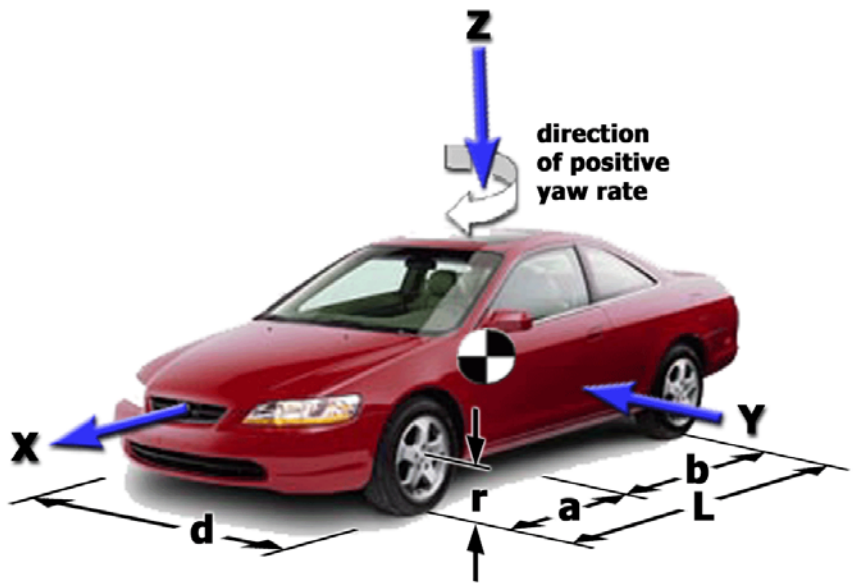


Figure 2.2: Diagram of vehicle coordinates and measurements

With the above assumptions, a state-space model can be obtained using the following methodology: first, the vehicle dynamics are written in state-space form with the tire forces acting as inputs to the system. We then solve for the tire forces as functions of the vehicle's states and control inputs. We then substitute these expressions into the state-space form to derive a state-space representation of the system. As a sign convention, the Society of Automotive Engineers standard coordinate system convention is used with the z-axis pointing into the road surface. The wheel torque that tends to spin the vehicle in the positive yaw direction, shown in Figure 2.2, will be considered positive. The resulting transfer function and state-space descriptions are based on the bicycle model with the states:

$$x \equiv \begin{bmatrix} y & \frac{dy}{dt} & \psi & \frac{d\psi}{dt} \end{bmatrix}^T \qquad (2.1)$$

and front steering input, $u \equiv \delta_f$, as the sole control channel. The transfer function from the traditional bicycle model is (in error coordinates):

$$\frac{y(s)}{\delta_f(s)} = \frac{1}{s^2} \cdot \frac{\frac{C_{\alpha f}}{m} \cdot s^2 + \frac{b \cdot L \cdot C_{\alpha f} \cdot C_{\alpha r}}{m \cdot I_z \cdot U} \cdot s + \frac{L \cdot C_{\alpha f} \cdot C_{\alpha r}}{m \cdot I_z}}{s^2 + \left(\frac{C_{\alpha f} + C_{\alpha r}}{mU} + \frac{a^2 \cdot C_{\alpha f} + b^2 \cdot C_{\alpha r}}{I_z \cdot U} \right) \cdot s + \frac{L^2 \cdot C_{\alpha f} \cdot C_{\alpha r}}{m \cdot I_z \cdot U^2} - \frac{a \cdot C_{\alpha f} - b \cdot C_{\alpha r}}{I_z}} \qquad (2.2)$$

The state space representation (in path error coordinates) is given in the standard form:

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t)$$
$$\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t) \tag{2.3}$$

with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{C_{\alpha f} + C_{\alpha r}}{mU} & \frac{C_{\alpha f} + C_{\alpha r}}{m} & \frac{b \cdot C_{\alpha r} - a \cdot C_{\alpha f}}{mU} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b \cdot C_{\alpha r} - a \cdot C_{\alpha f}}{I_z \cdot U} & \frac{a \cdot C_{\alpha f} - b \cdot C_{\alpha r}}{I_z} & -\frac{a^2 \cdot C_{\alpha f} + b^2 \cdot C_{\alpha r}}{I_z \cdot U} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{C_{\alpha f}}{m} \\ 0 \\ \frac{a \cdot C_{\alpha f}}{I_z} \end{bmatrix} \tag{2.4}$$

In these equations, the states have the following physical meaning:

y = lateral position of the vehicle on the road, measured from centerline

ψ = the yaw angle of the vehicle, measured with respect to the road

The control inputs are given by:

δ_f = the steering angle of the front tires

δ_r = the steering angle of the rear tires

ΔT = a torque moment applied by differential braking or unbalanced traction inputs

The model is dependent on the following parameters:

m = the vehicle's mass

L = the length of the vehicle

U = longitudinal velocity, assumed to be approximately constant

$C_{\alpha f}$, $C_{\alpha r}$ = front, rear wheel cornering stiffness, described further in the text below

a, b = distance from the C.G. to the front and rear axle, respectively

I_z = the moment of inertia of the vehicle about the z-axis

The resulting linear state-space model agrees with published dynamics from (Cho and Kim, 1995), among others. The expression of the system dynamics in state-space form reveals that the pole locations of the system are invariant with respect to the method of steering input. In general, most traditional vehicle systems utilize only one steering input acting from the angle of the front tires.

2.4 Model Fit of the Research Vehicles

Several vehicles are currently in use on the IRS, each with different operating capabilities. They range from a simple two-wheel-drive (2WD) front steering vehicle to a four-wheel-steer (4WS) vehicle with independent drive motors for each wheel as shown in Figure 2.3. The vehicle used in this experimental study, shown below, has separate DC motors mounted at each of the four wheels and has front and rear steering capability.

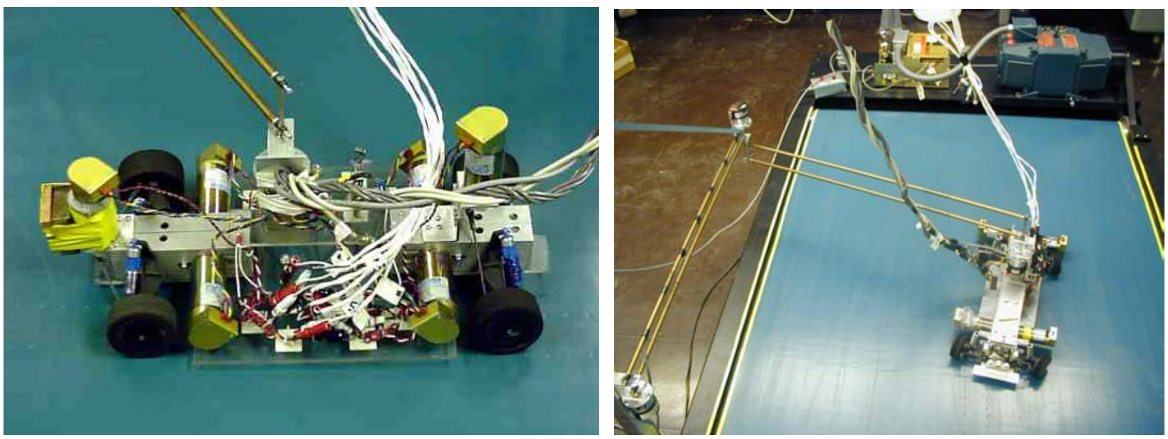


Figure 2.3: Pictures of the test vehicle

2.4.1 Parameter-Based Model Fit

Note that the parameters in the equations of Section 2.3 are all experimentally measurable, such as vehicle speed, mass, and moment of inertia. If these values are measured and substituted into the transfer functions listed above, a reasonable approximation of the vehicle's dynamics should be obtained. Although the measurement of the vehicle mass and lengths is trivial, the measurement of the cornering stiffnesses and z-moment is not obvious; details on this procedure are given in the author's M.S. thesis (Brennan, 1999). A summary of measured parameters is given in Table 2.1 below for various IRS testing vehicles.

Table 2.1: Measured parameter values for IRS research vehicles

Vehicle	M (kg)	Iz (kg-m^2)	a (m)	b (m)	Cαf (N/rad)	Cαr (N/rad)	U_{scale} (m/s)
Uberquad 2002	5.451	0.1615	0.146	0.219	65	110	3.0
Uberquad 2001	6.02	0.15	0.137	0.220	39	60	4.0
Uberquad 2000	6.02	0.153	0.137	0.222	40	52	1.98
Uberquad 1999	6.52	0.1830	0.155	0.235	96	65	3.0
4WS4WD 1999	4.02	0.1300	0.139	0.189	20	45	3.0
Uberquad 1998	2.66	0.0524	0.195	0.195	23	23	1.2
4WS4WD 1998	2.61	0.0656	0.15	0.185	117	97	1.2
2WS2WD 1997	1.47	0.0236	0.13	0.15	46	46	1.2

2.4.2 Frequency-Domain Fit

The bicycle and actuator dynamic models just developed will serve as the basis for the discussion in the following section regarding vehicle dynamic similitude. Naturally, the similitude development is only valid as long as the theoretical bicycle model dynamics are a valid description of vehicle dynamics for both scale and full-sized vehicles. For scale vehicles, we can compare the frequency responses when the vehicle was made to follow swept-sine inputs. Figure 2.4 below shows the frequency responses of the entire vehicle from driver input to yaw rate where the driver is steering the vehicle via front steering inputs and differential torque steering (torque steering is on the right). Overlaid on these plots are the bode plots resulting from the bicycle model and actuator dynamics using the bicycle parameters measured offline. For full-sized vehicles, extensive testing has compared the simple 2 DOF front-steering bicycle model to nonlinear 12 DOF simulations and actual vehicle tests.

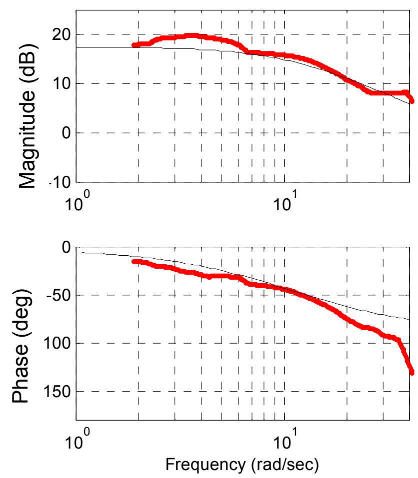


Figure 2.4: Frequency response fits from front steering to yaw rate.

It must be noted that the vehicle system dynamics from steering input to lateral position are open-loop unstable (in the sense of exponentially stable) due to two free integrators. Thus, the frequency responses above were obtained by "backing out" the linear responses from steering input to yaw rate while the vehicle was under autonomous control and being forced to track a sinusoid reference lateral position. Thus, frequency response modeling errors are expected at both low and high frequencies. Low-frequency modeling errors are due to the closed-loop controller attempting to compensate for slowly varying road disturbances. High-frequency modeling errors are naturally expected due to unmodeled system dynamics.

2.4.3 Time-Domain Fit

The following plot shows examples of time-domain validation of the model fit, in this case the showing the open-loop yaw-rate signal (which is generally the most difficult to match) versus an open-loop model fit and a desired closed-loop response (the higher-amplitude smooth response). The controller for this response is turned off to verify open-loop model dynamics.

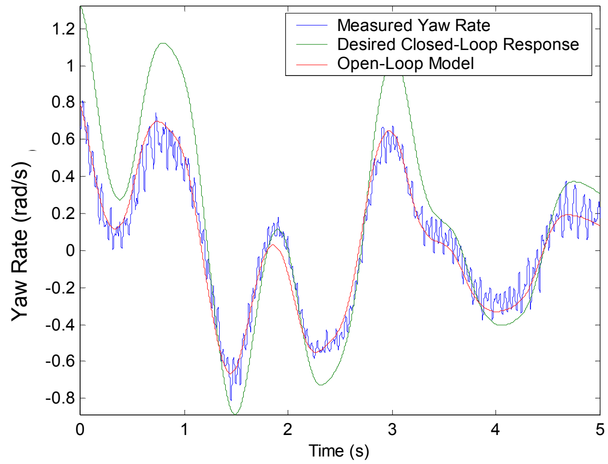


Figure 2.5: Time-domain responses showing model fit

By slightly tuning transfer function coefficients, the frequency domain and time domain results could be improved significantly. However, these results are presented 'as-is' to validate the direct use of the off line parameter identification. The results also provide confidence in the linear vehicle model used in the following discussion of dynamic similitude between the IRS scaled vehicles and full-size vehicles.

2.5 Contributions of this Chapter

The primary contributions of this chapter are as follows:

(1) Introduce vehicle notation and governing dynamic models for chassis motion at highway speeds.

(2) Illustrate that size-independent controller designs and model comparisons are necessary for certain control problems.

2.6 References

1. Bekker, M. G. Introduction to Terrain-Vehicle Systems. Ann Arbor: The University of Michigan Press, 1969.

2. Brennan, S., and A. Alleyne. "The Illinois Roadway Simulator: A Mechatronic Testbed for Vehicle Dynamics and Control." IEEE/ASME Transactions on Vehicle Dynamics and Control (1998).

3. Brennan, S., and A. Alleyne. "Robust Stabilization of a Generalized Vehicle, a Nondimensional Approach." 5th International Symposium on Advanced Vehicle Control (AVEC): OmniPress, 2000.

4. Brennan, S., and A. Alleyne. "Robust Scalable Vehicle Control Via Non-Dimensional Vehicle Dynamics." IEEE Journal of Vehicle System Dynamics (2001).

5. Brennan, Sean . "Modeling and Control Issues Associated With Scaled Vehicles." Masters. University of Illinois at Urbana-Champaign, 1999. Mechanical and Industrial Engineering, advisor: Dr. Alleyne

6. Brennan, Sean , and Andrew Alleyne. "Driver Assisted Yaw Rate Control." American Controls Conference: 1999. 1697-704.

7. Brennan, Sean , and Andrew Alleyne. "H-Infinity Vehicle Control Using NonDimensional Perturbation Measures." American Control Conference: OmniPress, 2002.

8. Cho, Young H. , and J. Kim. "Design of Optimal Four-Wheel Steering System." Vehicle System Dynamics 24 (1995): 661-82.

9. Dugoff, H., P. S. Fancher, and L. Segel. "An Analysis of Tire Traction Properties and Their Influence on Vehicle Dynamic Performance." SAE Transactions 79.SAE Paper No 700377 (1970): 341-66.

10. Furukawa, Moshimi, and Masato. "Advanced Chassis Control Systems for Vehicle Handling and Active Safety." Vehicle System Dynamics 28 (1997): 59-86.

11. Hatipoglu, Cem, Keith Redmill, and Umit Ozguner. "Automated Lane Change: Theory and Practice." Advances in Automotive Control 1998, Proceedings of the 2nd IFAC Workshop: G. Rizzoni and V. Utkin. Pergamon, 1926. 73-78.

12. LeBlanc, David J., et al. "CAPC: A Road-Departure Prevention System." IEEE Control Systems Magazine December (1996): 61-71.

13. Matsumoto, N., and M. Tomizuka. "Vehicle Lateral Velocity and Yaw Rate Control With Two Independent Control Inputs." ASME Journal of Dynamic Systems, Measurement, and Control 114.December (1992): 606-13.

14. McMahon, Thomas A., and John Tyler Bonner. On Size and Life. New York: Scientific American Books, Inc., 1983.

15. Nagai, Masao , Yutaka Hirano, and Sachiko Yamanaka. "Integrated Control of Active Rear Steering and Direct Yaw Moment Control." Vehicle System Dynamics 27 (1997): 357-70.

16. Pilutti, Tom , Galip Ulsoy, and Davor Hrovat. "Vehicle Steering Intervention Through Differential Braking." Proceedings of the American Control Conference: 1995. 1667-71.

17. Sampei, M., et al. "Arbitrary Path Tracking Control of Articulated Vehicles Using Nonlinear Control Theory." IEEE Trans. on Control System Technology 3.1 (1995): 125-31.

18. Shibahata, Y., K. Shimada, and T. Tomari. "Improvement of Vehicle Maneuverability by Direct Yaw Moment Control." Vehicle System Dynamics 22 (1993): 465-81.

19. Tomizuka, Masayoshi, and Karl J. Hedrick. "Advanced Control Methods for Automotive Applications." Vehicle System Dynamics 24 (1995): 449-68.

Chapter 3

The Mathematics of Dimensions

> I have often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude. It happens not infrequently that results in the form of "laws" are put forward as novelties on the basis of elaborate experiments, which might have been predicted *a priori* after a few minutes of consideration.
>
> Lord Raleigh

The intent of this chapter is to provide a mathematical framework of dimensional notation for the remainder of this thesis. Additionally, this chapter presents several of the basic dimensional scaling theories. The chapter begins with historical references to the subject of dimensions. The concept and definition of "dimension" is introduced. Based on these concepts, the Pi Theorem is presented and proved, and simple examples are provided. The theorem is developed upon the completeness of using a set of dimensionless parameters to describe physical equations. Examples are presented throughout this chapter, primarily for those who have not had much experience with dimensional analysis, and additional examples are given in the appendix. The familiar reader can skip or skim many of these without loss of continuity, as the focus is more on notational development of the subject.

3.1 History of Dimensional Analysis

Literature reviews on the subject of dimensional analysis generally appear in the first chapters of every book on the subject. However, one of the most concise and complete overviews of the topic can be found in a paper by Enzio Macagno (Macagno, 1971). Because of the inherent brevity of a journal-length historical review, this paper misses some of the more subtle historical points; these are emphasized in the discussion below. Regretfully, no discussion of the history of any topic can be truly complete, so the following should be accepted as a broad overview rather than a historically critical and complete summary.

The birth and usage of dimensional analysis began as a method to resolve the conflict between 'pure' and 'applied' mathematics. The concept of 'pure' mathematics likely began with Plato over 2000 years ago, who considered applied mathematics as lower than pure mathematics. A statement of 2 stones plus 3 stones is equal to 5 stones was seemingly not as 'pure' as the purely numerical statement of 2 plus 3 is equal to 5, simply because stones are destructible objects, impermanent, known only through sense experience, and imprecise in the sense that there are borderline cases of stones and non-stones. Most fields of physical analysis (including Control Theory) share a subtle bias in recorded history that pure mathematics is somehow higher than experimental work; see Chapter 1 of Ellis for a more complete discussion (Ellis, 1966). It is this disconnect that has caused some of the most notable of the recent mathematicians to resolve applied mathematics with pure theory by the introduction of 'units' associated with mathematical operations. Euler, Fourier, Maxwell, Heaviside, and many other of the brightest mathematicians did not discover or enforce a consistent use of dimensions for the sake of pure theory. Each recognized that mathematics in the purest form is useless without appropriate degrees of measure.

Galileo was one of the first to consider the nature of dimensions. In his *Dialogues concerning two new sciences*, he states (Focken, 1953) (p. 176):

> Clearly then, if one wishes to maintain in a great giant the same proportion of limb as that found in an ordinary man he must either find a harder and stronger material for making the bones, or he must admit a diminution of strength in comparison with men of medium stature; for if his weight be increased inordinately he will fall and be crushed under his own weight. Whereas, if the size of a body be diminished, the strength of that body is not diminished in the same proportion; indeed the smaller the body the greater its relative strength. Thus a small dog could probably carry on his back two or three dogs of his own size; but I believe that a horse could not carry even one of his own size.
>
> - Galileo of Galilee

An irony is that the concept of dimensional analysis is relatively young with regard to the concept of equations. As noted by Szirtes (Szirtes, 1997), the statement that dimensions must match in equations originates around 1765, when Leonhard Euler (1707-1783), discussed units and dimensional homogeneity in his book *Theoria motus corporum solidorum seu rigidorum.* Euler's considerations were primarily mathematical, and apart from his mathematical considerations, it does not appear that Euler appreciated the depth of meaning of dimensional statements within physical equations.

The 'fundamental' units known today as *length, mass,* and *time* were not always so fundamental, and according to Huntley they were formally named in 1832 by C. F. Guass (Huntley, 1952). However, it was Newton, in his *Principia* (II, proposition 32) who named three distinct entities as *length, inertia,* and *mass*. He used these quantities to define all remaining dimensions, which are often referred to as 'secondary' or 'derived' dimensions (or concepts, as Newton called them).

The first written consideration as to the universality of physical laws and the invariance of analysis with respect to scaling first began with Fourier according to the reviews of Bridgman and Focken (Focken, 1953; Bridgman, 1943)(p. 67). It was not until six decades after Euler that Jean Baptiste Joseph Fourier (1768-1830) established basic requirements for dimensional homogeneity in the last of his three versions of his work, *Theorie analytique de la chaleur,* or translated: *Theory of Heat*, 1822 (Focken, p. 128). In this work, Fourier used just two symbols to represent four different physical quantities. To correct the ambiguity without modifying the already-published equations, he described a computational method to determine the correct interpretation of each symbol. This consistency calculation incorporates exponents of dimensioned variables that directly precedes the work of Buckingham in the 1910's. Additionally, Fourier was the first to state that mathematical equations must be dimensionally homogenous and the argument of the exponential must be dimensionless (Hart, 1995).

As the 19th century progressed, the concept emerged that dimensions had their own algebra separate from the numeric calculations they govern. James Clerk Maxwell, famous in control theory for his differential analysis of the Watts governor, was quite influential in establishing this concept of an independent dimensional mathematics (Hart, page 7). It was for this reason that Maxwell introduced the square-bracket notation of representing dimensions so that such dimensional relationships could be expressed independent of variable quantities, and it is this notation that is so well-known today. Maxwell was a strong proponent of dimensioning magnetic and electric equations and including this analysis in science and engineering education. Unfortunately, Maxwell's formal analysis of dimensional mathematics only began in the later part of his life, culminating in a treatise on the topic in 1878. Maxwell died in 1879, and doubtless many of the theories developed a half-century later would have been discovered sooner had his studies been allowed to continue.

When Maxwell separated the algebra of dimensions from numerical analysis, a new research field of Dimensional Analysis quickly emerged. While it is difficult to date when the concept of analyzing problems based on dimensional algebra arose, Szirtes argues that it is clear that the concept began on or before 1899, when Lord Rayleigh applied dimensional analysis to the problem of the effect of temperature on the viscosity of a gas (Szirtes, 1997).

The most notable achievement in the area of dimensional analysis is the formulation of the Pi Theorem, stated by E. Buckingham in 1914 (Buckingham, 1914). Although Buckingham was initially given credit for the theorem (a credit still misattributed to this day), he later admitted that the algebraic form of his theorem was inspired by Riabouchinsky (Macagno, 1971). O'Rahilly notes that Riabouchinsky was not the first to discover the theorem, but that both Vaschy and Federmann had used their own form of the theorem earlier (O'Rahilly, 1939). Additionally, Jeans wrote a nearly equivalent form of the theorem in 1905. Despite these earlier works, Buckingham is appropriately universally credited with the first formal treatment of a method of dimensional analysis.

In Buckingham's approach, if one wishes to obtain a mathematical systems model, they first must establish what dimensional units are expected of each parameter within the system. From the dimensional algebra alone, Buckingham proved that it is possible to determine beforehand the necessary parameter groupings that will exist in the final system representation. These groupings greatly reduce the experimentation needed to generate a system model and a means of comparing disparate systems and experiments. It is on the basis of dimensional analysis and the Buckingham Pi theorem that much of the areas of Fluids and Heat Transfer were mathematically formalized. A deeper discussion of this theorem is given in more detail in the next section.

It is known from Einstein's work that he studied and considered dimensional analysis as a tool of discovery for physical laws (Isaacson and Isaacson, 1975; Bridgman, 1943). In particular, his work associated physical material properties to spectra, and the development of absorption/emission theory associated with solids appears to be based primarily on dimensional considerations. Unfortunately, his later (unsuccessful) work on unification theory appeared to ignore dimensional analysis; examples later illustrate this.

It should also be mentioned that the use of dimensional analysis is not necessarily uniform across all engineering fields. In order of usage, they might be listed as (with approximate dates in parenthesis):

- Civil Engineering (1850)
- Mechanical Engineering (1880)
- Physics (1880)
- Geological Engineering (1900)
- Chemical Engineering (1950)
- Biomechanical Engineering (1970)
- Biomedical Engineering (1980)
- Electrical and Computer Engineering (not used or taught at present date)

3.2 Common Notation

As noted by Langhaar, the general notation to 'extract' a dimension from a measurement is to use the bracket operator. This notation was originally introduced by Maxwell (Langhaar, 1951)(p 5). Thus, the dimension of a length variable in SI units would be given by:

$$[\text{length}] = \text{meter} \tag{3.1}$$

This bracket notation is *especially* cumbersome in attempting to denote the physical dimensions of a vector or a matrix, and not surprisingly these mathematical constructs suffer greatly from inappropriate addition and subtraction of dimensionally unlike quantities (Hart, 1995).

To prevent confusion, a different notation of dimensional and numerical equality is needed. This is especially true for this thesis which deals heavily with state-space, matrix, and vector representations. Within this thesis, if two quantities are dimensionally equal this special type of equality is denoted by the '$\underset{\smile}{=}$' operator. The under-curve was chosen to denote an underlying dimensional equality necessary for a higher-level numerical statement. Similarly, numerical equality apart from dimensional equality is implied by the '$\hat{=}$' operator. For example, the statement of Equation 3.2 below implies that a variable, V, has dimensions of apples, or

$V \mathrel{\underset{\smile}{=}} x\ \text{apples}$, where x can be any numerical number. In shorthand, we may write that $V \mathrel{\underset{\smile}{=}} \text{apples}$. Numerically, the statement also implies that V satisfies the numerical condition, $V \mathrel{\hat{=}} 2$. The combined dimensional and numerical equivalence is needed for complete equality.

$$V = 2\ \text{apples} \tag{3.2}$$

Specific consideration of equations violating one or more of these points is presented later.

Although it is a subtle point, it must be assumed that all measurements of a quantity are measured by quantities of similar kind and with a specified operation. The philosophical undertones of this can be found in the lengthy discussion in (Ellis, 1966), Ch. 1. That is, lengths are measured by a unit length placed end to end, masses by an inertial comparison to a unit mass, etc. The underlying reason for this is that the purpose of a dimension is to express how a measured quantity changes with respect to changes in the measurement units (Bridgman, 1943)(page 19). As noted by Duncan (Duncan, 1953)(p. 6):

> It cannot be too strongly emphasized that we never calculate with physical quantities but always with their measures, which are numbers. The symbols which appear in the mathematical equations related to a physical problem or process likewise represent numbers (measures, coefficients, indices), operations with numbers, or equality. To avoid circumlocution it is usual to say that such and such a symbol "is" the physical quantity in question, but it should never be forgotten that the symbol represents the number which is the measure of the physical quantity in terms of a selected unit.
>
> \- Duncan

In the common notation, there are three methods to express a dimensioned quantity: numeric format, symbolic format, and mixed format. Each is discussed in significant detail by Szirtes (Szirtes, 1997), but a summary is given below:

3.2.1 Numeric, Symbolic, and Mixed Formats

In the numeric format, a dimensioned variable is expressed as shown below:

$$\begin{aligned} \overbrace{\text{Age}}^{\text{name}} &= \overbrace{28}^{\text{magnitude}}\ \overbrace{\ }^{\text{space}}\ \overbrace{\text{years}}^{\text{dimension}} \\ \underbrace{A}_{\text{symbol of name}} &= \underbrace{28}_{\text{magnitude}}\ \underbrace{\ }_{\text{space}}\ \underbrace{\text{y}}_{\text{symbol of dimension}} \end{aligned} \tag{3.3}$$

In this example:

- The *name* may be a mnemonic string or symbol that represents the variable, but preferably one that does not go against common usage. For instance, do not use 'm' for age in a problem where mass may enter as a consideration. Additionally, the name should not be chosen using the dated practice of adding 'age' or 'ge' as a suffix to the dimension. The common practice of measuring distance in "mileage", area in "footage", or property in "acreage" makes no sense when other dimensions are considered. For instance, the question, "How old are you" becomes "what is your yearage", or "What is your salary" becomes "what is your dollarage". This misuse is slowly dying and is strongly discouraged.
- The *equality* (or inequality, as the case arises) always appears between the name and magnitude, and never between magnitude and dimension. For instance, "years = 28" is meaningless.
- The *magnitude* is a pure number or a symbol representing a single number.
- The *space* is required to prevent confusion with multiple variables. For instance, $A = 28\text{y}$ may imply that a variable 'A' is 28 times larger than another variable 'y'. Although the original usage of dimensions was in fact to show such a multiplicative relationship to a base dimension unit, this practice is no longer preferred.
- The *dimension* which identifies the measurement unit of the magnitude in addition to the method of measurement and the nature of the quantity. For instance:

$$\text{mass of Sean Brennan} = 82 \text{ kg} \tag{3.4}$$

  implies that the mass of one "kg" (whatever this is) must be multiplied by 82 to find the mass of Sean Brennan.

The symbolic format is the same as the numeric format except that the dimension field is missing. For instance,

$$d = \frac{1}{2} a \cdot t^2 \tag{3.5}$$

in which the distance, d, traveled by a free object is expressed in terms of an acceleration a, and a time period, t. The reason for the lack of dimension in such equations is that the dimension is already determined by the variables that are already expressed. For instance, expressing a in terms of 'm/s^2' and t in 's' forces dimension of 'm' for d. The "1/2" term is a pure number, and does not have any associated dimensions.

Mixed formats arise when certain known variables are substituted into a symbolic equation to produce an equation that is now fixed in dimensions but with some variables left in algebraic form. For instance, the volume of a box may be written as:

$$V = b \cdot w \cdot h \qquad (3.6)$$

But if the base and height are known to be 2 meters and 3 meters respectively, it may be written:

$$V = 6 \cdot w \text{ m}^3 \qquad (3.7)$$

However, this equation is only true if the width w is measured in meters, and therefore a dimension must be specified in the equation.

Confusion arises when this approach is generalized. For instance, with the falling object example, substituting 9.81 m/s^2 into the a term above gives the equation:

$$d = 4.905 \cdot t^2 \text{ m} \qquad (3.8)$$

However, it is unclear that the term t should be measured in seconds. A better expression would be:

$$d = 4.905 \cdot t^2 \text{ m}\left(/\text{sec}^2\right) \qquad (3.9)$$

However, no consensus has yet emerged on how to handle such situations. In practice, most equation writers incorrectly drop dimensions completely rather than confront this possible confusion. The result is generally a meaningless or useless equation, and the use of mixed dimensional formats for equations should be avoided if possible.

3.3 The Basic Concept of a Dimension

Different authors have their own definitions of a "dimension". Szirtes defines dimension as (Szirtes, 1997) (page 29):

> "A collection of previously agreed upon base quantities, joined by (possibly) repeated multiplication and division, but not addition or subtraction, which permits a numerical expression of any physical or abstract quantity so expressible." - Szirtes

Some authors, notably Ellis, have dedicated entire volumes to the consideration of what constitutes a dimension, measurement, and related topics. This level of detail quickly becomes largely philosophical, as typical engineering problems in control rarely if ever encounter questionable dimensioning systems.

One of the central concepts in dimension systems is the principle of the *Absolute Significance of Relative Magnitude*, first presented by Bridgman and discussed in detail by Isaacson (Isaacson and Isaacson, 1975) (p.4-6). The ideas behind this principle are simple: that the ratio of similar quantities in one dimensional system will remain the same as within any other dimensional system. For instance, if the density of a stone is 1.2 times the density of water, this ratio will be true regardless if density is measured in kg/m^3 or in lbs/gallon. Note that we use the semantic rather than mathematical meaning of similar. We now present the following theorem:

Theorem of the Absolute Significance of Relative Magnitudes: (due to Bridgman, presented below from Isaacson (Isaacson and Isaacson, 1975) (p. 5) *If the significance of relative magnitudes are absolute, i.e. invariant under dimensional transformations, then the units of measurement must be measured in terms of powers of the fundamental units.*

Proof: The following proof is implicitly derived from Euler's proof of his homogenous function theorem, which is presented in more detail in later sections. To understand the proof we must distinguish between fundamental and derived quantities. If a quantity S_i is fundamental (for instance, a meter), then it will always be a unit power of the fundamental unit (by definition). If the quantity is derived (for instance: density), then it must be measured by other quantities (for instance, kilograms-meters-seconds). The theorem states that the measure will always occur as powers of the fundamental units. To show this, we can always write the measurement, S_i, in the form: $S_1 = f\left(a_1, b_1, c_1\right)$, where the terms a_1, b_1, c_1 are measures along fundamental dimensions A,B, and C respectively. For instance, the density of a box 10 meters on a side weighing 1000 kg would be a function, $S_1 = f\left(1000, 10, 0\right)$ if the dimensions A, B, and C were kg, m, and s respectively.

The ratios of two measurements are then given as:

$$\frac{S_1}{S_2} = \frac{f(a_1, b_1, c_1)}{f(a_2, b_2, c_2)} \tag{3.10}$$

If the basic units of measure are changed so that they assume 1/x, 1/y, and 1/z of their original values, then the corresponding measurement along each of those dimensions is increased by x, y, and z respectively. In other words:

$$\frac{S_1}{S_2} = \frac{f(x \cdot a_1, y \cdot b_1, z \cdot c_1)}{f(x \cdot a_2, y \cdot b_2, z \cdot c_2)} \tag{3.11}$$

Because this ratio is invariant (by assumption), we may write:

$$\frac{f(a_1, b_1, c_1)}{f(a_2, b_2, c_2)} = \frac{f(x \cdot a_1, y \cdot b_1, z \cdot c_1)}{f(x \cdot a_2, y \cdot b_2, z \cdot c_2)} \tag{3.12}$$

or:

$$f(x \cdot a_1, y \cdot b_1, z \cdot c_1) = f(x \cdot a_2, y \cdot b_2, z \cdot c_2) \cdot \frac{f(a_1, b_1, c_1)}{f(a_2, b_2, c_2)} \tag{3.13}$$

We now differentiate partially with respect to *x*, and let f' denote the partial derivative of f with respect to the first term of f. Thus,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (x \cdot a_1)} \cdot \frac{\partial (x \cdot a_1)}{\partial x} = a_1 \cdot f' \tag{3.14}$$

we then can write:

$$a_1 \cdot f'(x \cdot a_1, y \cdot b_1, z \cdot c_1) = a_2 \cdot f'(x \cdot a_2, y \cdot b_2, z \cdot c_2) \cdot \frac{f(a_1, b_1, c_1)}{f(a_2, b_2, c_2)} \tag{3.15}$$

Since we can choose x, y, and z to be arbitrary scaling values, set x = y = z = 1, to obtain:

$$a_1 \cdot \frac{f'(a_1, b_1, c_1)}{f(a_1, b_1, c_1)} = a_2 \cdot \frac{f'(a_2, b_2, c_2)}{f(a_2, b_2, c_2)} \tag{3.16}$$

To investigate the nature of the function, we may fix the quantities a_2, b_2, c_2 while varying a_1, b_1, c_1. Thus, the right hand side remains constant, and the subscript may be dropped on the left-hand side:

$$\frac{a}{f(a,b,c)} \cdot \frac{\partial f(a,b,c)}{\partial a} = constant = A \qquad (3.17)$$

Integration gives:

$$f = S = k(b,c) \cdot a^A \qquad (3.18)$$

Here $k(b,c)$ is a constant dependent on only the second and third terms of f. We repeat the process for b and c to obtain:

$$f = S = k \cdot a^A \cdot b^B \cdot c^C \qquad (3.19)$$

with k independent of A, B, and C. Thus, the necessity of constant exponents is proved. We now examine whether constant exponents is sufficient to guarantee absolute significance of relative magnitudes. Again take the ratio:

$$\frac{S_1}{S_2} = \frac{f(x \cdot a_1, y \cdot b_1, z \cdot c_1)}{f(x \cdot a_2, y \cdot b_2, z \cdot c_2)} \qquad (3.20)$$

Since constant exponents are assumed:

$$\frac{S_1}{S_2} = \frac{a_1{}^A \cdot b_1{}^B \cdot c_1{}^C}{a_2{}^A \cdot b_2{}^B \cdot c_2{}^C} \qquad (3.21)$$

and assume again the units of measure are changed by 1/x, 1/y, and 1/z, so that:

$$\frac{S_1}{S_2} = \frac{(x \cdot a_1)^A (y \cdot b_1)^B (z \cdot c_1)^C}{(x \cdot a_2)^A (y \cdot b_2)^B (z \cdot c_2)^C} = \frac{a_1{}^A \cdot b_1{}^B \cdot c_1{}^C}{a_2{}^A \cdot b_2{}^B \cdot c_2{}^C} \cdot \frac{x^A \cdot y^B \cdot z^C}{x^A \cdot y^B \cdot z^C} \qquad (3.22)$$

And the last term on the right is unity; therefore, consistency of the relative magnitude is maintained. This completes the theorem.

A key assumption within any dimension system discussed in this thesis is that the principle of Absolute Significance of Relative Magnitude (ASRM) will be maintained

(Bridgman, 1943). Many common dimensional systems violate the assumption of ASRM, and therefore the dimensional analysis methods presented throughout this thesis will not apply (it might be argued that these are not dimensioning systems, but ordering systems). Very general examples of measurement systems that do not preserve ASRM are the pH scale of chemistry, the Celsius scale of temperature, the Brinell or Moh's scale of rock hardness, the F-scale of tornado classification, the Beaufort Scale of Wind Velocity, and the International Scale of Sea Roughness (Taylor, 1974). While non-ASRM systems can be used for inequality comparisons, quantities measured in such systems lose dimensional significance in mathematical manipulations previously defined. For instance, it is incorrect to consider a gypsum rock 2 times as hard as talc, simply because gypsum has a Moh's hardness of 2 and talc has a Moh's hardness of 1. An interesting case of a log scale that *does* preserve ASRM is the decibel scale of sound (Isaacson and Isaacson, 1975) (p. 7).

Definition: *A dimensional system of a problem will be called* **complete** *if it is comprised of sufficient fundamental dimensions to describe the magnitude of any numerically expressible quantity within the problem scope*.

Note that we cannot state that a dimensional system is in general 'complete' because completeness is only defined within a specific scope of a specific problem. A general statement of completeness is not provable, since it is impossible to prove that the dimensions of any unit system will be sufficient to describe *every* physical phenomenon that *can be* discovered, either today or in the far future. It is the intent of this definition to recognize that the concept of a dimension depends on the problem at hand, and specifically upon what equations are applied within the problem model.

It is therefore not surprising that a multitude of dimension systems have existed in the past. Interestingly, there has been a recent convergence to the SI system, but one might argue that this convergence is based more on a economic trend toward globalism rather than superior scientific and engineering utility of this particular system. The four most common systems of dimensions, the SI system, the MKS Force system, the British force system, and the British mass system are described in the appendix.

Each system has two types of units: *fundamental* and *derived* as discussed earlier. Fundamental units are standard measurements that are simply agreed upon by convention. As such, they may be arbitrary but for practical reasons they are generally chosen to fall into the domain of everyday human experience. Derived units are based on the usage of fundamental units in common measurements or formulas, and only appear as multiplications or divisions of fundamental units. For instance, a meter is a fundamental unit of length and a second is a fundamental unit of time, while velocity is usually measured in a derived unit of a meter per second. Whether a unit is fundamental or derived depends on the chosen dimensional system; force is a derived quantity in the SI system but a fundamental quantity in the MKS system (see appendix). As shown later in this chapter, there is no mathematical justification for why one vector basis is more fundamental than another. The difference in dimensional systems corresponds mathematically to a change of dimensional basis for a matrix representation of dimensional constraints, and thus is it is difficult to justify why one dimensional system may be superior to another.

3.4 Conversion Between Dimensional Systems

The need for dimensional mathematics usually first arises in converting measurements between different dimensional systems. Such conversions are in their essence based on an assumption of ASRM for each dimensional system. However, even for such simple conversions, constraints on mathematical operations begin to arise. In later sections, we investigate dimensional constraints on mathematical operations in a more general context, and then seek to develop mathematical problem representations that incorporate such constraints in an appropriate mathematical framework. Common conversion ratios are provided in the appendix between the various dimensional systems and many of the units within these (and others). Given these conversion factors, there are in general two methods to transform dimensions between different systems: fractional transformations and the use of dimensional exponents.

The most intuitive method to transform dimensions is to multiply a quantity repeatedly by one until the desired dimensions are obtained, a method outlined by Focken (Focken, 1953)(p. 8). An example illustrates this point. Let us assume that we want to convert 5 kg/m^3 into a quantity measured in terms of lb/ft^3. We begin with the original quantity and multiply it by conversions, each of which represents equivalence to unity:

$$\begin{aligned} 5\ \frac{\text{kg}}{\text{m}^3} &= 5\ \frac{\text{kg}}{\text{m}^3}\cdot \overbrace{\frac{1\cdot \text{m}^3}{(3.281)^3\ \text{ft}^3}}^{=1}\cdot \overbrace{\frac{2.205\ \text{lb}}{1\ \text{kg}}}^{=1} \\ &= 0.3121\ \frac{\text{lb}}{\text{ft}^3} \end{aligned} \tag{3.23}$$

Note that the units on the right hand side all cancel, and that each ratio on the right other than the original quantity are all equal to one. To perform the conversion, we noted that 3.281 ft = 1 m, and 2.205 lb = 1 kg. The advantage of the fractional method is that only a few of such conversion factors must be memorized to enable conversions between nearly all dimensional systems.

The method of conversion by dimensional exponents is formulated as follows: *Given a dimensional quantity x in dimensional system 1, what is the corresponding numerical value X in dimensional system 2?* To solve this, we assume that all dimensioned quantities obey the principle of ASRM previously discussed. Therefore, any quantity may be represented by basis dimensions in two different dimensioning systems as:

$$x\cdot \text{d}_1^{\,e_1}\cdot \text{d}_2^{\,e_2}\ldots \text{d}_n^{\,e_n} = X\cdot \text{D}_1^{\,e_1}\cdot \text{D}_2^{\,e_2}\ldots \text{D}_n^{\,e_n} \tag{3.24}$$

where:

$$\begin{aligned} \text{d}_1,\text{d}_2\ldots &= \text{fundamental dimensions in system 1} \\ \text{D}_1,\text{D}_2\ldots &= \text{fundamental dimensions in system 2} \\ e_1,e_2\ldots &= \text{exponents of dimensions in both systems} \\ n &= \text{number of dimensions in each system} \\ X &= \text{desired numerical factor} \end{aligned}$$

The goal is to find *X*. Note that the exponents, $e_1,\ldots,e_n$, are the same on both sides of the equation, a condition that is relaxed and discussed in more detail later. We also note that the dimensions $\text{d}_1,\text{d}_2\ldots$ correspond to dimensions $\text{D}_1,\text{D}_2\ldots$; if d_I represents a length, then D_I must also represent a length. Their magnitudes may differ, but they must represent the same type of physical dimensions. We can then write in general:

$$
\begin{aligned}
\mathrm{d}_1 &= k_1 \cdot \mathrm{D}_1 \\
\mathrm{d}_2 &= k_2 \cdot \mathrm{D}_2 \\
&\vdots \\
\mathrm{d}_n &= k_n \cdot \mathrm{D}_n
\end{aligned}
\tag{3.25}
$$

Upon substitution into the above equation, this yields:

$$
x \cdot \left(k_1 \cdot \mathrm{D}_1\right)^{e_1} \cdot \left(k_2 \cdot \mathrm{D}_2\right)^{e_2} \ldots \left(k_n \cdot \mathrm{D}_n\right)^{e_n} = X \cdot \mathrm{D}_1^{\ e_1} \cdot \mathrm{D}_2^{\ e_2} \ldots \mathrm{D}_n^{\ e_n} \tag{3.26}
$$

By inspection, the numerical equality of both sides requires:

$$
x \cdot k_1^{\ e_1} \cdot k_2^{\ e_2} \ldots k_n^{\ e_n} = X \tag{3.27}
$$

This allows us to generally solve any dimensional transformation.

We illustrate this method by again solving the previous example. We write 5 kg/m^3 as:

$$
5 \cdot \mathrm{kg}^1 \cdot \mathrm{m}^{-3} = X \cdot k_1^{\ 1} \cdot k_2^{\ -3} \tag{3.28}
$$

with $x = 5$, $\mathrm{d}_1 = \mathrm{kg}$, $\mathrm{d}_2 = \mathrm{m}$, $e_1 = 1$, and $e_2 = \text{-}3$. There are two conversion equations that provide k_1 and k_2:

$$
\begin{aligned}
1\ \mathrm{kg} &= 2.205\ \mathrm{lb} \\
1\ \mathrm{m} &= 3.281\ \mathrm{ft}
\end{aligned}
\tag{3.29}
$$

and thus $k_1 = 2.205$ and $k_2 = 3.281$. Finally,

$$
\begin{aligned}
&\overbrace{\left(5\ \frac{\mathrm{kg}}{\mathrm{m}^3}\right)}^{x} \cdot \overbrace{\left(2.205\ \frac{\mathrm{lb}}{\mathrm{kg}}\right)^{1}}^{k_1^{e_1}} \cdot \overbrace{\left(3.281 \frac{\mathrm{ft}}{\mathrm{m}}\right)^{-3}}^{k_2^{e_2}} = X \\
&X = 0.3121\ \frac{\mathrm{lb}}{\mathrm{ft}^3}
\end{aligned}
\tag{3.30}
$$

Which is the same answer found previously. Numerically, it should be obvious that both methods are identical and based fundamentally on the assumption of ASRM, but that the second method is more conducive to algebraic analysis. Therefore, this second method is used in the remainder of this thesis.

In the case where dimensional basis vectors are not aligned between two systems, the conversions may be problematic. For instance, the units of mass in the metric MKS system are

$m^{-1} \cdot kg \cdot s^2$, but in the metric SI system it is simply m. We therefore require considerations of situations where fundamental dimensions may be distinct in one system but coupled in another, a problem first discussed by Bridgman (Bridgman, 1943) (p 32). Given a unit dependent on three fundamental dimensions, what is the unit representation in a different set of dimensions where the fundamental dimensions are not the same? The following analysis considers only 3 dimensions, but extends to an arbitrary number of dimensions. Mathematically, the problem is formulated as:

$$\begin{aligned} d_1^{a_1} \cdot d_2^{a_2} \cdot d_3^{a_3} &= k_1 \cdot D_1 \\ d_1^{b_1} \cdot d_2^{b_2} \cdot d_3^{b_3} &= k_2 \cdot D_2 \\ d_1^{c_1} \cdot d_2^{c_2} \cdot d_3^{c_3} &= k_3 \cdot D_3 \end{aligned} \qquad (3.31)$$

The goal is to solve for the values of the terms $d_1 ... d_3$ in terms of $D_1 ... D_3$. To solve the equation, we take the logarithm of both sides:

$$\begin{aligned} a_1 \cdot \log(d_1) + a_2 \cdot \log(d_2) + a_3 \cdot \log(d_3) &= \log(k_1 \cdot D_1) \\ b_1 \cdot \log(d_1) + b_2 \cdot \log(d_2) + b_3 \cdot \log(d_3) &= \log(k_2 \cdot D_2) \\ c_1 \cdot \log(d_1) + c_2 \cdot \log(d_2) + c_3 \cdot \log(d_3) &= \log(k_3 \cdot D_3) \end{aligned} \qquad (3.32)$$

Which is linear algebraic in the exponents:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} \log(d_1) \\ \log(d_2) \\ \log(d_3) \end{bmatrix} = \begin{bmatrix} \log(k_1 \cdot D_1) \\ \log(k_2 \cdot D_2) \\ \log(k_3 \cdot D_3) \end{bmatrix} \qquad (3.33)$$

The solution to this equation is:

$$\begin{bmatrix} \log(d_1) \\ \log(d_2) \\ \log(d_3) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \log(k_1 \cdot D_1) \\ \log(k_2 \cdot D_2) \\ \log(k_3 \cdot D_3) \end{bmatrix} \qquad (3.34)$$

or:

$$
\begin{aligned}
d_1 &= (k_1 \cdot D_1)^{e_{11}} \cdot (k_2 \cdot D_2)^{e_{12}} \cdot (k_3 \cdot D_3)^{e_{13}} \\
d_2 &= (k_1 \cdot D_1)^{e_{21}} \cdot (k_2 \cdot D_2)^{e_{22}} \cdot (k_3 \cdot D_3)^{e_{23}} \\
d_3 &= (k_1 \cdot D_1)^{e_{31}} \cdot (k_2 \cdot D_2)^{e_{32}} \cdot (k_3 \cdot D_3)^{e_{33}}
\end{aligned} \qquad (3.35)
$$

where the exponents, $e_{11} \ldots e_{33}$, correspond to the elements of the inverse matrix above. The existence of the inverse of the matrix above is guaranteed by conditions of dimensional usage discussed later. An example is given in Bridgman expressing the quantity of 15 tons-mass·miles/hour in new units of H.P., ft/sec, and ergs (Bridgman, 1943).

3.5 Dimensional Constraints on Mathematics

The following two sections deal directly with the mathematics associated with dimensions. This first section assumes that the dimensions of the arguments to a function constrain the allowable mathematical operations within the function. Therefore it seeks to develop constraints on the set of allowable mathematical operations for a given set of dimensioned quantities. The second section reverses the approach, and assumes that mathematical operations can be redefined in a way that dimensionally constrains the arguments. Therefore, the second section seeks to develop constraints on dimensioned quantities that enter as arguments for a given set of dimensionally true mathematical operations. The mathematical constraints assumed in the second section are specifically chosen to yield results dimensionally consistent with the first section. Both sections are unified so that mathematical operations on dimensioned quantities appear to assume a vector-like dependence on the dimensions of the arguments.

3.5.1 Functional Homogeneity: Constraints on Addition, Subtraction, and Inequalities

Before dealing with the most basic constraints on mathematical operations, we first introduce a special set of parameters whose dimension is unity, called 'dimensionless' parameters. Unfortunately, the term 'dimensionless' is a misnomer, as these parameters do have a dimension: i.e. a dimension of unity! As several authors note (namely Szirtes), one would not say that the x^2 term in Equation 3.36 is "coefficient-less", or that the right-most term is "exponent-less" (Szirtes, 1997).

$$y = 2 \cdot x^3 + x^2 + 6 \cdot x \qquad (3.36)$$

However, the term "dimensionless" has become popular in usage, and to prevent confusion it will be grudgingly used without further comment in the remainder of this thesis.

Definition (Dimensional Homogeneity): *A set of variables, parameters, measurements, etc. are* **Dimensionally Homogenous** *if every member of the set has equivalent dimensions* (Note: this is a modification of a definition originally presented by Langhaar (Langhaar, 1951) (p. 49).

For example, the following set is dimensionally homogenous: $\{2 \text{ apples}, 3 \text{ apples}, 5 \text{ apples}\}$ but the following is not: $\{2 \text{ apples}, 3 \text{ oranges}, 5 \text{ pears}\}$

In a physical equation, dimensionally unlike quantities cannot be added, subtracted, or be regarded as greater than, less than, or equal to each other. Velocities can be added to velocities, but densities cannot be added to velocities. One can say that 2 meters is less than 5 meters, but to say that 2 meters is less than, equal to, or greater than 4 seconds is meaningless. The addition of 2 meters plus 2 inches is technically valid, only because a conversion factor may generate a common unit of addition along the length dimension. Therefore, addition, subtraction, or any type of equality or inequality is only defined over arguments that all belong to a dimensionally homogenous set. For instance, Equation 3.37 is dimensionally and numerically valid:

$$2 \text{ apples} + 3 \text{ apples} = 5 \text{ apples} \qquad (3.37)$$

But Equation 3.38 is not valid dimensionally:

$$2 \text{ apples} + 3 \text{ oranges} \not\underset{\smile}{=} 5 \text{ pears} \qquad (3.38)$$

And Equation 3.39 is not valid numerically:

$$2 \text{ meters} + 3 \text{ meters} \not\overset{\frown}{=} 4 \text{ meters} \qquad (3.39)$$

We state without proof that if $a \underset{\smile}{=} b$ and $b \underset{\smile}{=} c$, then $a \underset{\smile}{=} c$. The following definition formally specifies conditions for equality and inequality.

Definition (Correctness): *An equation or inequality is correct if and only if the two sides maintain numerical equality and dimensional homogeneity.*

Corollary: *The number 0 (zero) may have any dimension.* This is obvious from two different homogenous equations, each with different dimensions. One can always 'solve for zero' by rearrangement of any equation.

A common example of incorrect dimensional usage are the addition of Newton's Law and the Law of Thermodynamics, shown in Equation 3.40:

$$\begin{aligned} F &= m \cdot a && \text{(numerically correct, homogenous)} \\ Q_{in} - Q_{out} &= W && \text{(numerically correct, homogenous)} \\ F + Q_{in} - Q_{out} &= m \cdot a + W && \text{(numerically correct, not homogenous)} \end{aligned} \qquad (3.40)$$

This equation is numerically correct, but can never describe a physical phenomenon because the addition of the equations violates the assumptions of dimensional homogeneity. For this reason, such equations are sometimes called *nugatory* (Duncan, 1953) (p. 42).

3.5.2 Dimensions of Products

Theorem of Dimensioned Products: (this is the form presented in Szirtes, but this theorem was first stated by Fourier around 1830, and later by Langhaar in 1950 (Szirtes, 1997; Langhaar, 1951) (p. 51) *The product of the dimensions of two variables is the dimension of the product of the two variables.*

$$(V_1)\cdot(V_2) \mathrel{\underset{\smile}{=}} (V_1 \cdot V_2) \qquad (3.41)$$

Proof: Using the format previously presented for dimension quantities, denote:

$$\begin{aligned} V_1 &= m_1 \cdot \mathrm{d}_1 \\ V_2 &= m_2 \cdot \mathrm{d}_2 \end{aligned} \qquad (3.42)$$

where m_1 and m_2 are the magnitudes, and d_1 and d_2 are the dimensions of variables V_1 and V_2 respectively. By definition, $V \mathrel{\underset{\smile}{=}} \mathrm{d}_1$ and $V_2 \mathrel{\underset{\smile}{=}} \mathrm{d}_2$. Therefore:

$$(V_1)\cdot(V_2)\doteqdot d_1\cdot d_2 \tag{3.43}$$

By direct multiplication, $V_1\cdot V_2 = m_1\cdot m_2\cdot d_1\cdot d_2$, so:

$$(V_1\cdot V_2)\doteqdot d_1\cdot d_2 \doteqdot (V_1)\cdot(V_2)$$

This completes the theorem. The theorem is general in that it can be extended to any number of variables.

Corollary: *The product of the dimensions of a set of* n *variables is the dimension of the product of this set of* n *variables. That is:*

$$(V_1)\cdot(V_2)\ldots(V_n)\doteqdot(V_1\cdot V_2\ldots V_n) \tag{3.44}$$

Corollary: *The dimension of a power of a variable is the power of the dimension of that variable. That is:*

$$(V)^n\doteqdot(V^n) \tag{3.45}$$

We therefore note that scalar multiplication is defined for any set of dimensioned variables.

3.5.3 Dimensions of Quotients

Theorem of Dimensioned Quotients: (again, this is the form presented in Szirtes, but this theorem was first stated by Fourier around 1830 (Szirtes, 1997)) *The quotient of the dimensions of two variables is the dimension of the quotient of the two variables.*

$$\frac{(V_1)}{(V_2)}\doteqdot\left(\frac{V_1}{V_2}\right) \tag{3.46}$$

Proof: Again we prove this directly using the format previously presented for dimension quantities, denote V_1 and V_2 as before:

$$\frac{(V_1)}{(V_2)}\doteqdot\frac{d_1}{d_2} \tag{3.47}$$

By direct division, $V_1 / V_2 = m_1 / m_2 \cdot \mathrm{d}_1/\mathrm{d}_2$, so:

$$\left(\frac{V_1}{V_2}\right) \mp \frac{\mathrm{d}_1}{\mathrm{d}_2} \mp \frac{(V_1)}{(V_2)}$$

This completes the theorem. As in the case of multiplication, we find that scalar division is also defined for any set of dimensioned variables.

3.5.4 Dimensions of Associations

Theorem of Dimensioned Associations: (this is the form presented in Szirtes) *If* d_1*,* d_2*, and* d_3 *are the dimensions of* V_1*,* V_2*, and* V_3 *respectively, then* $(\mathrm{d}_1 \cdot \mathrm{d}_2) \cdot \mathrm{d}_3 \mp \mathrm{d}_1 \cdot (\mathrm{d}_2 \cdot \mathrm{d}_3)$.

Proof: (direct proof). Define the variables, $V_{12} \equiv V_1 \cdot V_2$ and $V_{23} \equiv V_2 \cdot V_3$.

$$(V_{12}) \cdot (V_3) \mp (\mathrm{d}_1 \cdot \mathrm{d}_2) \cdot \mathrm{d}_3 \mp (V_1 \cdot V_2) \cdot (V_3) \mp (V_1) \cdot (V_2) \cdot (V_3) \qquad (3.48)$$

$$(V_1) \cdot (V_2 \cdot V_3) \mp \mathrm{d}_1 \cdot (\mathrm{d}_2 \cdot \mathrm{d}_3) \mp (V_1) \cdot (V_2 \cdot V_3) \mp (V_1) \cdot (V_2) \cdot (V_3) \qquad (3.49)$$

And therefore:

$$(\mathrm{d}_1 \cdot \mathrm{d}_2) \cdot \mathrm{d}_3 \mp \mathrm{d}_1 \cdot (\mathrm{d}_2 \cdot \mathrm{d}_3)$$

This completes the theorem. Associative multiplications are therefore defined over any set of permissible multiplications.

3.5.5 Dimensions of Differentials

Theorem of Dimensioned Differentials: (again, this is the form presented in Szirtes) *The dimension of the quotient of differentials of variables* V_1 *and* V_2 *is the quotient of dimensions* V_1 *and* V_2 .

$$\frac{dV_1}{dV_2} \mp \frac{V_1}{V_2} \qquad (3.50)$$

Proof: (direct proof) Using the definition of the derivative of a function, using V_1 and V_2 as the variables:

$$\frac{dV_1}{dV_2} \triangleq \lim_{\Delta V_2 \to 0} \frac{\Delta V_1}{\Delta V_2} \tag{3.51}$$

where ΔV_1 and ΔV_2 are both restricted to be finite. By the Theorem of dimensional quotients:

$$\frac{dV_1}{dV_2} \mathrel{\overline{\overline{\smile}}} \frac{\Delta V_1}{\Delta V_2} \mathrel{\overline{\overline{\smile}}} \frac{(\Delta V_1)}{(\Delta V_2)} \mathrel{\overline{\overline{\smile}}} \frac{V_1}{V_2}$$

This completes the theorem. Therefore differentiation is constrained over the same set as quotients, which is the set of all dimensioned scalars.

Theorem of Dimensioned *n*

Theorem of Dimensioned n^{th}-order Differentials: (from Szirtes) *The dimension of the* n$^{\text{th}}$ *order differential of variable V_1 with respect to V_2 is the quotient of dimensions V_1 and $V_2^{\,n}$.*

$$\frac{d^nV_1}{dV_2^{\,n}} \mathrel{\overline{\overline{\smile}}} \frac{V_1}{V_2^{\,n}} \tag{3.52}$$

Proof: (note that the proof in Szirtes is incorrect in that it does not prove the general case) (Proof by induction) The case for $n=1$ was given above. We now show that if the theorem is true for order *n*, it is true for order *n+1*. Define $V_3 = \frac{d^nV_1}{dV_2^{\,n}}$, so by assumption, $V_3 \mathrel{\overline{\overline{\smile}}} \frac{V_1}{V_2^{\,n}}$

$$\frac{dV_3}{dV_2} \triangleq \lim_{\Delta V_2 \to 0} \frac{\Delta V_3}{\Delta V_2} \tag{3.53}$$

where ΔV_3 and ΔV_2 are both restricted to be finite. By the Theorem of dimensional quotients:

$$\frac{dV_3}{dV_2} \mathrel{\overline{\overline{\smile}}} \frac{\Delta V_3}{\Delta V_2} \mathrel{\overline{\overline{\smile}}} \frac{(\Delta V_3)}{(\Delta V_2)} \mathrel{\overline{\overline{\smile}}} \frac{(V_1)}{(V_2)\cdot(V_2)^n} \mathrel{\overline{\overline{\smile}}} \frac{V_1}{V_2^{\,n+1}}$$

This completes the theorem. Again, these are the same constraints as observed with quotients of dimensioned quantities.

3.5.6 Dimensions of Integrals

Theorem of Dimensioned Integrals: (from Szirtes) *The dimension of an integral is the product of the dimensions of the integrand and the dimension of the independent variable. Thus, if:*

$$I = \int y(x) \cdot dx \qquad (3.54)$$

$$I \mathrel{\overline{\overline{\mp}}} y(x) \cdot dx \mathrel{\overline{\overline{\mp}}} \left(y(x)\right) \cdot (dx) \mathrel{\overline{\overline{\mp}}} y(x) \cdot x \qquad (3.55)$$

Proof: The proof is trivial since the integration is of the form of a product (much as a differentiation is the form of a quotient). The proof of dimensioned products was provided earlier.

Theorem of Multiple Dimensioned Integrals: (from Szirtes) *The dimension of a multiple integral is the product of the dimension of the integrand and the dimension of all the independent variables. Thus, if:*

$$I = \int_{(1)} \int_{(2)} \cdots \int_{(n)} y(x_1, x_2 \ldots, x_n) \cdot dx_1 \cdot dx_2 \ldots \cdot dx_n \qquad (3.56)$$

then

$$\begin{aligned} I &\mathrel{\overline{\overline{\mp}}} y(x_1, x_2 \ldots, x_n) \cdot dx_1 \cdot dx_2 \ldots \cdot dx_n \\ &\mathrel{\overline{\overline{\mp}}} \left(y(x_1, x_2 \ldots, x_n)\right) \cdot (dx_1) \cdot (dx_2) \ldots \cdot (dx_n) \\ &\mathrel{\overline{\overline{\mp}}} \left(y(x_1, x_2 \ldots, x_n)\right) \cdot x_1 \cdot x_2 \ldots \cdot x_n \end{aligned} \qquad (3.57)$$

Proof: The proof by induction follows that for differentials provided earlier.

Integrations and multiple integrations are therefore defined over any set of permissible multiplications, which is again the set of all dimensioned scalars, variables, etc.

3.5.7 Dimensions of Transcendental Functions

Corollary: *The results of exponents and/or any transcendental function are only defined if the arguments are dimensionless.* This follows from the Taylor-series expansion of any transcendental function, which would generate a summation of terms of different dimensions for any case other than when the argument is dimensionless. The constraints of addition/subtraction then require equivalent units, which is only possible for dimensionless arguments.

Theorem of Gage Invariance: (originally proposed by Euler) *If an equation is correct in one dimensional system, i.e. dimensionally homogenous and numerically correct, then it is also correct in any other dimensional system consistency applied.*

Proof: (from Szirtes) We may consider an equation of three variables and two dimensions, but the proof extends without loss to an arbitrary number of variables and dimensions. Additionally, this proof considers only a one-term equation, but the analysis also extends to functions of an arbitrary number of terms. This is evident by the constraint of dimensional homogeneity, which requires any additional terms to share a similar dimension (and hence the analysis below) as the one term analyzed here.

Define the governing equation as:

$$V_1 = Q \cdot V_2^{q_2} \cdot V_3^{q_3} \tag{3.58}$$

where $V_1 \ldots V_3$ are the variables, Q is a non-zero number, and q_2, q_3 are exponents. Without loss of generality, assume that there are only two fundamental dimensions: d_1, d_2. Then if $m_1 \ldots m_3$ are magnitudes, we can write $V_1 \ldots V_3$ as:

$$\begin{aligned} V_1 &= m_1 \cdot \mathrm{d}_1^{e_{11}} \cdot \mathrm{d}_2^{e_{12}} \\ V_2 &= m_2 \cdot \mathrm{d}_1^{e_{21}} \cdot \mathrm{d}_2^{e_{22}} \\ V_3 &= m_3 \cdot \mathrm{d}_1^{e_{31}} \cdot \mathrm{d}_2^{e_{32}} \end{aligned} \tag{3.59}$$

where $e_{11} \ldots e_{32}$ are dimensional exponents. Therefore,

$$V_1 = m_1 \cdot \mathrm{d}_1^{e_{11}} \cdot \mathrm{d}_2^{e_{12}} = \left(Q \cdot m_2^{q_2} \cdot m_3^{q_3}\right) \cdot \mathrm{d}_1^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot \mathrm{d}_2^{e_{22} \cdot q_2 + e_{32} \cdot q_3} \tag{3.60}$$

By the *numerical correctness* assumption of the equation in the statement of the theorem:

$$m_1 = Q \cdot m_2^{q_2} \cdot m_3^{q_3} \tag{3.61}$$

By the *dimensional homogeneity* assumption of the theorem:

$$\begin{aligned} e_{11} &= e_{21} \cdot q_2 + e_{31} \cdot q_3 \\ e_{12} &= e_{22} \cdot q_2 + e_{32} \cdot q_3 \end{aligned} \tag{3.62}$$

Now we change the dimensional system. Define,

$$\begin{aligned} \mathrm{d}_1 &= k_1 \cdot \mathrm{D}_1 \\ \mathrm{d}_2 &= k_2 \cdot \mathrm{D}_2 \end{aligned} \qquad (3.63)$$

From 3.64, the value of V_1 after the dimensional transformation is given as:

$$\begin{aligned} V_1 &= \left(Q \cdot m_2^{q_2} \cdot m_3^{q_3}\right) \cdot \left(k_1 \cdot \mathrm{D}_1\right)^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot \left(k_2 \cdot \mathrm{D}_2\right)^{e_{22} \cdot q_2 + e_{32} \cdot q_3} \\ &= \left(Q \cdot m_2^{q_2} \cdot m_3^{q_3} \cdot k_1^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot k_2^{e_{22} \cdot q_2 + e_{32} \cdot q_3}\right) \cdot \mathrm{D}_1^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot \mathrm{D}_2^{e_{22} \cdot q_2 + e_{32} \cdot q_3} \end{aligned} \qquad (3.65)$$

As an alternative, we can transform V_1 directly from Equation 3.58 as:

$$V_1 = m_1 \cdot \left(k_1 \cdot \mathrm{D}_1\right)^{e_{11}} \cdot \left(k_2 \cdot \mathrm{D}_2\right)^{e_{12}} \qquad (3.66)$$

Thus, the left hand side of the original equation becomes, upon substitution of the constraints of 3.62 and 3.63 :

$$V_1 = \left(Q \cdot m_2^{q_2} \cdot m_3^{q_3} \cdot k_1^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot k_2^{e_{22} \cdot q_2 + e_{32} \cdot q_3}\right) \cdot \mathrm{D}_1^{e_{21} \cdot q_2 + e_{31} \cdot q_3} \cdot \mathrm{D}_2^{e_{22} \cdot q_2 + e_{32} \cdot q_3}$$

This completes the theorem. This result is equivalent to Equation 3.65. Thus, both the dimensions and numerical correctness of the equation are preserved under dimensional transformations.

Example: Dimensional homogeneity of the mass-spring-damper

In Chapter 1 the mass-spring-damper was introduced with a governing dynamics equation given by:

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + k \cdot x = F(t) \qquad (3.67)$$

and parameters:

$$m = 1\ kg, \beta = 1\ \frac{kg}{sec}, k = 1\ \frac{kg}{sec^2} \qquad (3.68)$$

The differential equation in the kilogram-meter-second system is:

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}+1=F(t) \tag{3.69}$$

We show that the form of this equation is invariant under a dimensional transformation of time, length, and mass. Examining the original system, if we allow a transformation in length, time, and mass given by:

$$\begin{pmatrix} l \rightarrow l^* \mid l^* = L \cdot l \\ t \rightarrow t^* \mid t^* = T \cdot t \\ m \rightarrow m^* \mid m^* = M \cdot t \end{pmatrix} \tag{3.70}$$

The coefficients *and derivatives* must be modified as:

$$\begin{aligned} & m=\frac{1}{M}\ , \quad \beta=\frac{T}{M}, \quad k=\frac{T^2}{M}, \\ & F=\frac{T^2}{ML}, \quad \frac{d^2x}{dt^2}=\frac{T^2}{L}\frac{d^2x^*}{dt^{*2}}, \quad \frac{dx}{dt}=\frac{T}{L}\frac{dx^*}{dt^*}, \\ & x=\frac{x^*}{L} \end{aligned} \tag{3.71}$$

So that the new system becomes:

$$\frac{d^2x^*}{dt^{*2}}+\frac{dx^*}{dt^*}+1=F(t^*) \tag{3.72}$$

Which is identical to the original system.

Corollary: *In any analytically derived equation, both sides of the equation must have identical dimensions. All numbers appearing in this equation must be dimensionless constants.*

This statement is obvious from the assumption that the derivation is correct, and from the fact that in any analytically derived equation, all constants are dimensionless. The following are also obvious results.

Corollary a: (from Szirtes) *If a quantity is dimensionless in a dimensional system, then it is also dimensionless in all other dimensional systems.*

Corollary b: (from Szirtes) *The magnitude of a dimensionless variable is invariant upon the dimensional system in which the constituents of that variable are expressed.*

Corollary c: *The only variables that are invariant upon dimensional transformations are dimensionless variables.*

Proof of (a)-(c): (direct proof, the first part was presented in Szirtes) Given a dimensionless variable, π_i, measured with fundamental dimensions, $\mathrm{d}_1 \ldots \mathrm{d}_n$, it can always be written as:

$$\pi_i = Q \cdot (\mathrm{d}_1)^0 \cdot (\mathrm{d}_2)^0 \ldots \cdot (\mathrm{d}_n)^0 \qquad (3.73)$$

where Q is the magnitude of the variable. The proof of (a) and (b) is seen under a transformation to a different dimensional system. We define the relations:

$$\begin{aligned} \mathrm{d}_1 &= k_1 \cdot \mathrm{D}_1 \\ \mathrm{d}_2 &= k_2 \cdot \mathrm{D}_2 \\ &\vdots \\ \mathrm{d}_n &= k_n \cdot \mathrm{D}_n \end{aligned} \qquad (3.74)$$

So that:

$$\pi_i = Q \cdot (k_1 \cdot \mathrm{D}_1)^0 \cdot (k_2 \cdot \mathrm{D}_2)^0 \ldots \cdot (k_n \cdot \mathrm{D}_n)^0 \qquad (3.75)$$

$$\pi_i = Q \cdot (\mathrm{D}_1)^0 \cdot (\mathrm{D}_2)^0 \ldots \cdot (\mathrm{D}_n)^0$$

This completes the theorem for parts (a) and (b). The proof of (c) follows from the fact that for any dimensioned quantity, one of the dimensional exponents must be nonzero. Thus, one of the dimensional exponents of this quantity raised to a power as in Equation 3.75 would have a non-zero exponent. Thus, the equivalency of magnitude would be violated.

The above discussion should serve to demonstrate that constraining mathematical operations over a given set of dimensioned arguments is a tedious method to approach the problem. In most cases, clearly the operations are allowable independent of dimensional considerations, yet for addition, subtraction, inequalities, exponents, and transcendental functions, particular assumptions on the arguments were required. Mathematics as a tool of analysis is generally not conducted in the manner above, where each operation itself is critiqued for validity. Rather, general sets of operational constraints are defined broadly over sets of permissible arguments. It should be clear that some implicit assumption of dimensional algebra is being assumed in many of the above operations. Rather than relying on post-operational verification on a case-by-case basis, many have argued (notably Hart) that it would be better to define mathematics in a manner that includes dimensional considerations over a field of dimensioned quantities. This is the approach presented in the next section, which serves as a summary of the main points of Hart's work on the subject (Hart, 1995).

3.6 Mathematical Constraints on Dimensional Arguments

A routine method for dealing with dimensioned quantities in numeric operations is to substitute them into expressions derived for real or complex numbers. However, the field of real numbers is, by definition, closed under addition while dimensioned quantities are not. For instance, if '1 meter' is a real number, then '1 meter^2' is also a real number, and the addition $(1\text{ m})+(1\text{ m}^2)$ would be a valid operation (this argument was first presented formally by Hart, p. 20-21, but was inferred by Euler). Clearly the extension of real number methods to dimensioned quantities is incorrect.

The incompatibility between properties of real numbers and properties of dimensioned measurements is made clear by an example. Consider the statement of Equation 3.76, which is valid for all real numbers. This statement is meaningless for a dimensioned measurement, say x = 1 meter.

$$x^2 + x \geq -\frac{1}{4}, \qquad (3.76)$$

Indeed, any argument beginning with a statement like "*Let x be a real number…*" and ending with a conclusion *"…and the distance is less than 1 meter*." should be suspect (Hart, 1995) (p. 24). The incorrect usage of measurements in the form of the example above is in practice prevented by an implicit inclusion of dimensional algebra into the procedures of real analysis. Psychologically, scientists and engineers already use a richer mathematics than explicitly acknowledged in current, formal mathematical systems.

However, problems arise for vector and matrix operations because it becomes exceedingly difficult to check dimensional results. In general, the lack of dimensional constraints in matrix mathematics often leads to contradictions. For example, the quantities $\mathbf{A}^2$ and $\mathbf{A}+\mathbf{A}^2$ are both defined for the following matrix of real values:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.77}$$

However, for the following three dimensioned matrices, i.e. matrices with dimensioned scalars as their elements (Hart, 1995) (p. 10, 32):

$$\mathbf{X} = \begin{bmatrix} 1\ \text{m} & 1\ \text{s} \\ 1\ \text{s} & 1\ \text{m} \end{bmatrix},\ \mathbf{Y} = \begin{bmatrix} 1\ \text{m} & 1\ \text{m}\cdot\text{s} \\ 1\ \text{m/s} & 1\ \text{m} \end{bmatrix},\ \mathbf{Z} = \begin{bmatrix} 1 & 1\ \text{s} \\ 1\ \text{s}^{-1} & 1 \end{bmatrix} \tag{3.78}$$

the quantities: $\mathbf{X}^2$ and $\mathbf{Y}+\mathbf{Y}^2$ are both undefined, while the terms $\mathbf{Y}^2, \mathbf{Z}^2$, and $\mathbf{Z}+\mathbf{Z}^2$ are defined. Clearly one does not want to check every matrix operation for dimensional consistency, but rather to determine dimensional conditions for the arguments of matrix functions for which these operations are dimensionally defined.

3.6.1 The Dimensional Basis Vector

Over the past several decades, there has been a growing consideration of dimensions as vector quantities; even 50 years ago, there was a very noticeable dimensional flavor to Laghaar's presentation of dimensional analysis (Langhaar, 1951). This notion probably originated as an examination of dimensions as a directed graphs, which were used to reveal the relationship between dimensions and vectors (Isaacson and Isaacson, 1975) (p16-18). The figure below shows the dimensional graph structure of several dimensioned quantities:

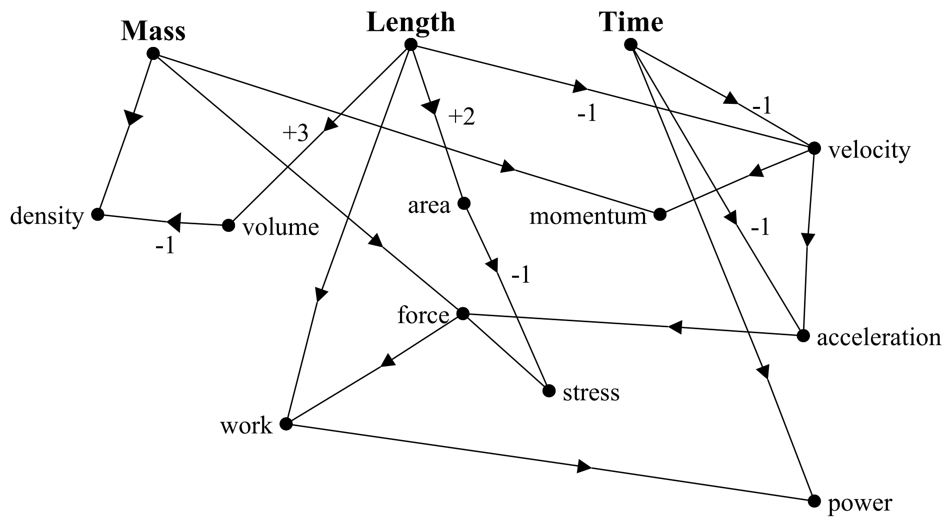


Figure 3.1: Directed graph structure of dimensional quantities

A more intuitive representation would be to use mass, length, and time as axes, with the dimensional exponents explicitly as ordinates for vectors. For instance, velocity as a vector is shown below:

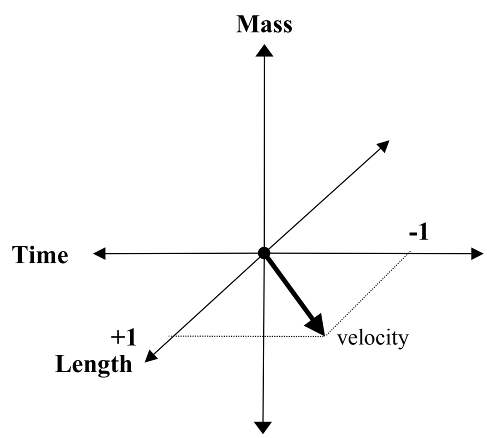


Figure 3.2: Diagram form of a dimensional basis

Some notion of the inherent structure of homogenous equations can be seen with a simple example. If we consider the dimensions associated with the velocity of a falling object traversing a given distance:

$$v = a \cdot t \tag{3.79}$$

We obtain the following graph:

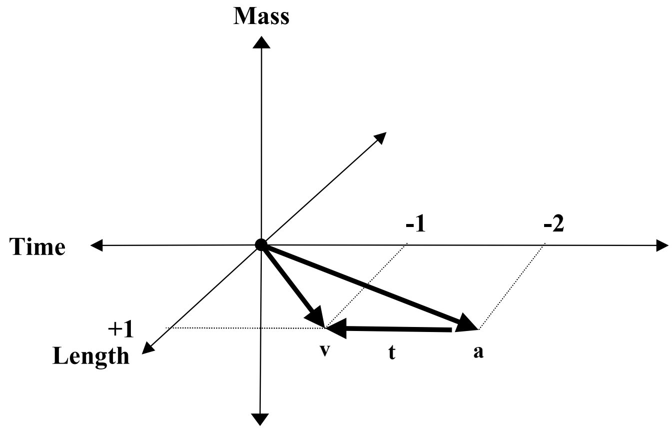


Figure 3.3: Dimensional basis graph for velocity calculation

An interesting fact about the above graph is that the equation forms a closed loop. It should be clear that *all dimensionally homogenous equations map to closed loops in dimensional basis graphs.* This important result will serve as the geometric insight to dimensional analysis approaches presented later, and the reader is referred to Hart (Hart, 1995) (p. 51) for a better diagram of dimensional representations.

It is obviously useful to consider the notion of *dimensional basis vectors*, based on the concept of a *d*-dimensional affine space, where *d* is the number of fundamental dimensions. By arranging dimensional exponents in this particular form, dimensional operations are greatly simplified and can be generalized to matrix concepts such as basis, span, independence, and rank in later sections.

To form the dimensional basis vector, we first assume that there are a fixed number of fundamental dimensions, $F_1, F_2, ..., F_m$. The dimensions of the parameters can be written in powers of these fundamental dimensions:

$$\begin{aligned} [x] &= F_1^{a_1} \cdot F_2^{a_2} \dots \cdot F_m^{a_m} \\ &= \prod_{i=1}^{m} F_i^{a_i} \end{aligned} \qquad (3.80)$$

The dimensional basis vector would then be written as:

$$\begin{array}{c|c} & x \\ \hline F_1 & a_1 \\ F_2 & a_2 \\ \vdots & \vdots \\ F_m & a_m \end{array} \tag{3.81}$$

or may be denoted in shorthand by the notation:

$$[\![x]\!] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \tag{3.82}$$

Note that the value, order, and number of the elements in the dimensional basis vector depend on the chosen basis for the dimension system. We now define the meaning of a dimensioned scalar:

Definition: (Hart, 1995)*A* **dimensioned scalar** *is an ordered pair,* $(\mathrm{r,v})$*, consisting of a real number and a vector.*

Example: Vehicle velocity measurements

The dimensions of a vehicle velocity term may be written in the international standard SI (meters-kilograms-second) system as the following:

$$velocity \mathrel{\overline{\overline{\smash{\mathrm{\,}}}}} \mathrm{meters}^{1} \cdot \mathrm{kilograms}^{0} \cdot \mathrm{seconds}^{-1} \tag{3.83}$$

Or in dimensional basis forms as:

$$\begin{array}{c|c} & velocity \\ \hline \text{meters} & 1 \\ \text{kilograms} & 0 \\ \text{seconds} & -1 \end{array} \tag{3.84}$$

So:

$$[\![velocity]\!] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{3.85}$$

Thus, a velocity of 20 m/s would be recorded as $\left(20, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$.

Under an ordered-pair definition, the operations of addition, subtraction, multiplication, and exponentiations are given by (Hart, 1995) (p. 33):

$$(\mathrm{r}_1,\mathrm{v}_1) \pm (\mathrm{r}_2,\mathrm{v}_2) = \begin{cases} (\mathrm{r}_1 \pm \mathrm{r}_2, \mathrm{v}_1), & \text{if } \mathrm{v}_1 = \mathrm{v}_2; \\ \text{undefined, if } \mathrm{v}_1 \neq \mathrm{v}_2 \end{cases} \tag{3.86}$$

$$(\mathrm{r}_1,\mathrm{v}_1) \cdot (\mathrm{r}_2,\mathrm{v}_2) = (\mathrm{r}_1 \cdot \mathrm{r}_2, \mathrm{v}_1 + \mathrm{v}_2) \tag{3.87}$$

$$(\mathrm{r}_1,\mathrm{v}_1) / (\mathrm{r}_2,\mathrm{v}_2) = \begin{cases} (\mathrm{r}_1 / \mathrm{r}_2, \mathrm{v}_1 - \mathrm{v}_2), & \text{if } r_2 \neq 0; \\ \text{undefined, if } r_2 = 0 \end{cases} \tag{3.88}$$

$$(\mathrm{r}_1,\mathrm{v}_1)^{(\mathrm{r}_2,\mathrm{v}_2)} = \begin{cases} (\mathrm{r}_1^{\mathrm{r}_2}, \mathrm{r}_2 \cdot \mathrm{v}_1), & \text{if } \mathrm{v}_2 = 0, \text{ and } \mathrm{r}_1^{\mathrm{r}_2} \text{ and } \mathrm{r}_2 \cdot \mathrm{v}_1 \text{ are defined;} \\ \text{undefined, otherwise.} \end{cases} \tag{3.89}$$

The first statement required dimensional homogeneity as mentioned in the previous section. The constraint on division prohibits zero of any dimension. The constraints on exponentiation are that the exponent be dimensionless, and that the result remains defined in the appropriate fields. For instance, $\mathrm{r}_1^{\mathrm{r}_2}$ must be defined to eliminate square roots of negative numbers if the field is real numbers, and $\mathrm{r}_2 \cdot v_1$ must be defined to ensure the dimensional basis vector remains in the proper class of real, rational, or integer to prevent statements like $y = 2.4\ \mathrm{m}^{3\pi}$.

Hart has argued that the above constraints are best unified by a logarithmic representation of dimensioned operations. For instance, he defines the basis logarithm as:

Definition. *Given an n-component dimensional basis,* $B=\left(B_1,B_2,\ldots,B_m\right)$*, and a dimensioned scalar,* x*, in the span of the basis, its* **dimensional logarithm***, denoted by* $\log_B\left(x\right)$*, is the n-vector of exponents,* $\left(e_1,e_2,\ldots,e_m\right)$ *, which makes* $\prod_{i=1}^{m} B_i^{e_i}$ *have the same dimensions as x.*

On comparison, this is the same as the dimensional basis defined earlier. However, Hart's notation generalizes the basic properties of conventional logarithms in many respects. Specifically,

$$\log_B\left(a\cdot b\right)=\log_B\left(a\right)+\log_B\left(b\right) \tag{3.90}$$

but the sum is a vector sum. For dimensionless b,

$$\log_B\left(a^b\right)=\mathrm{b}\cdot\log_B\left(a\right) \tag{3.91}$$

but the product is a scalar product. As a difference between traditional logarithms, the dimensional logarithm treats its argument in a sign-symmetric manner, and is undefined if x is not in the span of B (Hart, 1995).

Therefore, in dimensional basis form, we note that standard dimensioned multiplication and division map to addition and subtraction (respectively) of the corresponding dimensional subspace vectors. Positive or negative powers of a dimensioned variable correspond to multiplication or division (respectively) of a dimensional subspace vector. These results directly correspond to the mathematical requirements encountered in the previous section. The example below illustrates these ideas:

Example: Density of a fluid

The density of an unknown fluid is given by Equation 3.92, where ρ is the fluid density, L is the length of a box containing the fluid, and m is the mass of the fluid in the box.

$$\rho=\frac{m}{L^3} \tag{3.92}$$

If we choose the dimension system as $\left[\text{meters}\quad\text{kilograms}\right]^T$, then the dimensional basis vectors are given by the table:

	m	L	ρ
mass	1	0	1
length	0	1	-3

(3.93)

It is clear that:

$$\begin{gathered}[\![\rho]\!] = 1\cdot[\![m]\!] - 3\cdot[\![L]\!] \\ \begin{bmatrix}1 & -3\end{bmatrix}^{\mathrm{T}} = 1\cdot\begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}} - 3\cdot\begin{bmatrix}0 & 1\end{bmatrix}^{\mathrm{T}}\end{gathered} \tag{3.94}$$

And the example works out as expected.

We note that the real numbers are isomorphically embedded in dimensional scalars as a case where the dimensional vector is zero. Additionally, there are an infinite number of unique zero dimensional vectors, because the number of dimensional basis vectors is problem-dependent. For instance, a kinematics problem measuring position and velocity of a linkage might have a dimensional basis of [meters, seconds] and the zero dimensional vector would be $\begin{bmatrix}0 & 0\end{bmatrix}$. Whereas, a dynamics problem might have a basis vector of [meters, seconds, kilograms] and the zero dimensional vector would be $\begin{bmatrix}0 & 0 & 0\end{bmatrix}$. To say that these two zero-vectors are equal is analogous to saying that two boxes have equivalent contents because one box has zero apples and another has zero oranges. It is the operation of measuring that determines the quantity, and it is simply incorrect to compare two systems of incompatible measurement systems even if the numerical values of the measurement are both zero.

3.6.2 Sign Symmetries of Physical Equations

Before extending the notion of dimensional mathematics above scalar concepts and more toward dimensioned vectors, we must first address the notion of symmetry. In the measure of a dimensioned quantity, the numeric quantity, r, in the pair (r,v) representing a measurement can be obtained by the definition of a dimensional basis: i.e. a ratio of an unknown quantity to a measurement standard. For instance, a measurement of 2.7 meters would be obtained by taking the ratio of an unknown length to that of a fixed unit of length, in this case one meter. By the ASRM principle (Absolute Significance of Relative Magnitude), we can state a specific principle for dimensioned quantities:

Definition: Basis-Independence Principle: *Calculated quantities are independent of the dimensional basis.* (Hart, 1995) (p. 42)

This is a restatement of the ASRM principle, and it implies that it does not matter if we measure a distance in units of meters, yards, light-years, etc., the result is the same length. This is so universally agreed upon for scalar systems that it is completely uncontroversial.

However, for vector quantities, the issue is quite different, and calculated quantities can take on seemingly arbitrary values depending on the chosen unit system. For instance, in control theory the measure of a state-magnitude is often used as a metric of performance. For instance, at a particular time-point a state may be defined as:

$$\mathbf{x} = \begin{bmatrix} 1 \text{ meter} \\ 1 \text{ second} \end{bmatrix} \qquad (3.95)$$

Formally, the vector is generally represented by taking only the real portion of the vector elements. Unfortunately, the real-component could be $[1 \quad 1]^T$, $[10 \quad 1]^T$, or $[0.001 \quad 1]^T$ depending on if meters, decimeters or kilometers were used as units. Similarly, the "magnitude" would 'vary' between $\sqrt{2}$, $\sqrt{101}$, and $\sqrt{1.000001}$. It should be clear that the magnitude operator is not defined over nearly all classes of dimensioned vectors.

One method of analyzing mathematical systems is by considering their automorphisms, i.e. the symmetries under which their axioms remain invariant. Hart notes that dimensional mathematics are preserved under scaling of all dimensional quantities, that is an equation remains true if one carefully doubles all lengths, quadruples all areas, etc. Thus, one obtains a set of gauge transformations that form an algebraic group of operations under which conclusions of dimensioned equations remain invariant (Hart, 1995)(p. 45). This is used to prove various versions of the pi-theorem presented later in this chapter.

From a less mathematical viewpoint, gauge transformations are equivalent to considering different units in the dimensional basis. The invariance of physical laws to such changes is simply a consequence of the principle of similitude and a consequence of the ASRM principle. Hart's important contribution, and one that is certainly unique, is that he considers the consequence of allowing scaling by negative quantities in the set of gauge transformations. This idea certainly isn't entirely new… it is well known in the field of physics that all closed-system physical laws remain valid if time is reversed. Hart argues that this symmetry is itself a confirmation of the ASRM principle, yet one that isn't usually considered for other primary dimensions such as length or mass. As a central argument, he notes that the principles of invariance upon which ASRM is based work equivalently well for negative measures, and the addition of a sign-asymmetry into theory is unnecessary (Hart, 1995).

An important consequence of this realization is that the algebra of dimensioned quantities is such that the truth of all physical equations is preserved under sign reversal of the fundamental units. As an important example, Benjamin Franklin, in the discovery of the nature of voltage and electrons, arbitrarily (but incorrectly) defined current flow such that the direction of current flow is *opposite* that of the direction of electron movement. To Franklin's credit, the electron was not discovered for nearly a century after his decision to label the polarity of a battery. However, if tomorrow society as a whole decided to change the sign convention on the unit of charge (Coulomb) to reflect the correct flow of electrons, then all equations would remain equally valid when the appropriate sign changes were made. Hart presents similar references regarding the 'arrow of time'. In the general case, we therefore require that no true physical equation distinguish 'positive' quantities from 'negative' ones. Such sign symmetry is an important issue for defining vector and matrix magnitudes.

3.6.3 Dimensional Notation and Basic Dimensional Algebra

The main function of matrix and vector representations is that they are concrete forms that can be manipulated to study the abstract notion of linear transformations. However, both dimensioned matrices and vectors will change certain properties under transformation between different dimensional systems and so a dimensionally study of both is necessary. A dimensioned vector is hereafter defined as an n-tuple in which each entry is a dimensioned scalar. Any dimensioned vector belongs to a *complete dimensioned vector space* defined as the set of vectors whose dimensional logarithms are equal but whose numerical components are arbitrary. We define a *subspace* of the dimensioned vector space in the traditional manner, as a subset that is closed under addition and scalar products by dimensionless scalars. Thus, a *dimensioned vector space* is one defined to be a complete vector space, or a subspace of one (Hart, 1995) (p. 62). In a similar manner, a dimensioned matrix is hereafter defined as an array with dimensioned scalars as its elements.

The following notational definitions are useful for discussions on dimensioned systems hereafter. The following definitions are based strongly on Hart (Hart, 1995) (p. 64), yet stronger definitions are presented below as they were found needed for implementation on actual problems. The term 'mathematical dimensions' is used to denote the size of the array, whereas the term 'physical dimensions' will denote the physical units of the elements of the vector or array.

$\mathbf{A},\mathbf{B},\ldots$	Dimensioned matrices
$\mathbf{a},\mathbf{b},\ldots$	Dimensioned vectors
$a,b,c,\ldots$	Dimensioned scalars
$\mathbf{A} \;\overline{=}\; \mathbf{B}$	$\mathbf{A}$ and $\mathbf{B}$ are dimensionally equal
$\mathbf{A} \;\overline{\neq}\; \mathbf{B}$	$\mathbf{A}$ and $\mathbf{B}$ are dimensionally unequal
$\mathbf{A} \sim \mathbf{B}$	$\mathbf{A}$ and $\mathbf{B}$ are dimensionally similar
$\mathbf{A} \approx \mathbf{B}$	$\mathbf{A}$ and $\mathbf{B}$ are dimensionally parallel
$m \times n$	Mathematical dimensions (size) of $\mathbf{A}$
i,j,k,l	Integer indices of a matrix
$\mathbf{A}_{ij},[\;]_{ij}$	Component of the matrix in i^{th} row, j^{th} column
$\mathbf{a}_i$	The i^{th} component of a vector
$\mathbf{A}^T$	The transpose of matrix $\mathbf{A}$
$\mathbf{A}^{[-1]}$	The dimensional inverse of matrix $\mathbf{A}$
$\mathbf{I}_\mathbf{a}$	Identity matrix for a dimensioned space defined by $\mathbf{a}$

As before, we define the $\overline{=}$ operator as extracting the dimensions of a scalar, vector, or array and the result of the operator is a scalar, vector, or array with equivalent dimensional components of the arguments. Therefore,

$$\text{If } \mathbf{A} = \begin{bmatrix} 1\text{ m} & 2\text{ s} \\ 3 & 4\text{ kg} \end{bmatrix}, \text{ then } \mathbf{A} \;\overline{=} \begin{bmatrix} \text{meters} & \text{seconds} \\ \text{unitless} & \text{kilograms} \end{bmatrix} \qquad (3.96)$$

Additionally, assuming that the comparisons are between objects of similar mathematical dimensions, we define scalar, vector, and matrix dimensional equality as:

$$\mathbf{A} \underset{\smile}{=} \mathbf{B} \text{ iff } [\mathbf{A}]_{ij} \underset{\smile}{=} [\mathbf{B}]_{ij} \;\; \forall i, j \qquad (3.97)$$

For instance,

$$\begin{bmatrix} 1\text{ m} & 2\text{ s} \\ 3 & 4\text{ kg} \end{bmatrix} \underset{\smile}{=} \begin{bmatrix} 6\text{ m} & 2.4\text{ s} \\ 2 & 9\text{ kg} \end{bmatrix} \qquad (3.98)$$

A less restrictive form is used by Hart, where dimensional similarity is used to denote unit systems that are a unitless scaling factor away from dimensional equality. For instance, $[1\text{ meter}] \not\underset{\smile}{=} [2\text{ feet}]$, but $[1\text{ meter}] \sim [2\text{ feet}]$ because both meters and feet are length measurements and can be made dimensionally equal by a unitless scaling factor.

$$\mathbf{A} \sim \mathbf{B} \text{ iff } \exists\text{c}, \mathbf{A} \underset{\smile}{=} c \cdot \mathbf{B};\; c \underset{\smile}{=} 1. \qquad (3.99)$$

For instance,

$$\begin{bmatrix} 1\text{ m} & 2\text{ s} \\ 3 & 4\text{ kg} \end{bmatrix} \underset{\smile}{=} \begin{bmatrix} 6\text{ m} & 2.4\text{ s} \\ 2 & 9\text{ kg} \end{bmatrix} \qquad (3.100)$$

Note also that:

$$\mathbf{A} \underset{\smile}{=} \mathbf{B} \Rightarrow \mathbf{A} \sim \mathbf{B} \qquad (3.101)$$

This statement is only true in one direction. The third relation, whether two quantities are dimensionally parallel, is weaker than dimensional equality or dimensional similarity. It is defined to hold between two scalars, vectors, or matrices that differ by the same multiplicative dimensioned constant:

$$\mathbf{A} \approx \mathbf{B} \text{ iff } \exists\text{c}, \mathbf{A} \underset{\smile}{=} c \cdot \mathbf{B} \qquad (3.102)$$

This equation has no dimensional constraints on the constant, c. Thus, all scalars are dimensionally parallel, but not all dimensioned vectors or matrices are parallel. For instance, the matrices of Equation 3.103 are dimensionally parallel, but not dimensionally similar or dimensionally equal.

$$\begin{bmatrix} 1\text{ m} & 2\text{ s} \\ 3 & 4\text{ kg} \end{bmatrix} \approx \begin{bmatrix} 2\text{ m}\cdot\text{s} & 1\text{ s}^2 \\ 9\text{ s} & 7\text{ kg}\cdot\text{s} \end{bmatrix} \qquad (3.103)$$

Also note that:

$$\mathbf{A} \mathrel{\overline{\overline{=}}} \mathbf{B} \;\Rightarrow\; \mathbf{A} \sim \mathbf{B} \;\Rightarrow\; \mathbf{A} \approx \mathbf{B} \qquad (3.104)$$

Again, the converse does not hold. The following also hold for the $\mathrel{\overline{\overline{=}}}$, $\sim$, and $\approx$ relations (only the dimensionally equal case is shown):

(Reflexivity) $$\mathbf{A} \mathrel{\overline{\overline{=}}} \mathbf{A} \qquad (3.105)$$

(Symmetry) $$\mathbf{A} \mathrel{\overline{\overline{=}}} \mathbf{B} \Rightarrow \mathbf{B} \mathrel{\overline{\overline{=}}} \mathbf{A} \qquad (3.106)$$

(Transitivity) $$\mathbf{A} \mathrel{\overline{\overline{=}}} \mathbf{B} \text{ and } \mathbf{B} \mathrel{\overline{\overline{=}}} \mathbf{C} \;\Rightarrow\; \mathbf{A} \mathrel{\overline{\overline{=}}} \mathbf{C} \qquad (3.107)$$

Many mathematical situations arise where we require a matrix or vector to have the property where all elements have identical dimensions. Such vectors or matrices will be called **dimensionally uniform**. Therefore, using Hart's notation:

$$\begin{aligned} \mathbf{A} \text{ dim. uniform iff } & \mathbf{A}_{ij} \mathrel{\overline{\overline{=}}} \mathbf{A}_{kl} \;\forall i,j,k,l \\ \text{iff } & \mathbf{A} \approx \mathbf{B} \text{ for } \mathbf{B} \text{ dimensionless} \end{aligned} \qquad (3.108)$$

A special case of a dimensionally uniform matrix is one whose elements are all dimensionless.

We now consider the important concept of a **dimensional inversion** operator. Naturally, the multiplication of a matrix with its own dimensional inverse is intended to produce a unitless matrix. Only a small subset of matrices is multipliable, and these matrices share a special dimensional form. Based on later results that dimensionally constrain such multipliable matrices, the dimensional inverse is defined by forming the dimensional reciprocal element-by-element and then taking the transpose of the resulting matrix. Mathematically:

$$\left[\mathbf{A}^{[-1]}\right]_{ij} \mathrel{\overline{\overline{=}}} \frac{1}{\mathbf{A}_{ji}} \qquad (3.109)$$

Again, this notation is similar to Hart, but the requirements are stronger in that dimensional equality rather than dimensional similarity is required. Note that this operator is only applicable to dimensional relationships, and therefore it exists for all matrices. As an example:

$$\mathbf{A} = \begin{bmatrix} 3\ \dfrac{\text{volts}}{\text{apples}} & 6\ \dfrac{\text{volts}}{\text{pears}} \\ 2\ \dfrac{\text{amps}}{\text{apples}} & 4\ \dfrac{\text{amps}}{\text{pears}} \end{bmatrix} \qquad (3.110)$$

$$\mathbf{A}^{[-1]} \overset{=}{\sim} \begin{bmatrix} 1\ \dfrac{\text{apples}}{\text{volts}} & 1\ \dfrac{\text{apples}}{\text{amps}} \\ 2\ \dfrac{\text{pears}}{\text{volts}} & 2\ \dfrac{\text{pears}}{\text{amps}} \end{bmatrix} \qquad (3.111)$$

Dimensionally, $\mathbf{A} \cdot \mathbf{A}^{[-1]} \overset{=}{\sim} \mathbf{1}$. Note that the mathematical inverse may not be defined (for instance, the matrix in 3.111), but the dimensional inverse is always defined. Also note that the dimensional inverse is not unique. As another example, consider:

$$\mathbf{A} = \begin{bmatrix} 3\text{ apples} & 6\text{ grapes} \\ 2\text{ oranges} & 4\text{ pears} \end{bmatrix} \qquad (3.112)$$

$$\mathbf{A}^{[-1]} \overset{=}{\sim} \begin{bmatrix} 1\ \dfrac{1}{\text{apples}} & 1\ \dfrac{1}{\text{oranges}} \\ 0\ \dfrac{1}{\text{grapes}} & 1\ \dfrac{1}{\text{pears}} \end{bmatrix} \qquad (3.113)$$

This illustrative in that the left-hand matrix is not multipliable, i.e. no matrix can be multiplied by the left matrix and still produce a dimensionally valid result. Even though the dimensional and traditional inverse are defined, there is no possible $\mathbf{A}^{[-1]}$ such that $\mathbf{A} \cdot \mathbf{A}^{[-1]} \overset{=}{\sim} \mathbf{1}$ holds. The following relations also hold for the inverse operator:

$$\begin{gathered} \left(\mathbf{A}^{[-1]}\right)^{[-1]} \overset{=}{\sim} \mathbf{A} \\ \left(\mathbf{A}^{[-1]}\right)^{T} \overset{=}{\sim} \left(\mathbf{A}^{T}\right)^{[-1]} \\ \mathbf{A} \begin{pmatrix} \overset{=}{\sim} \\ \sim \\ \approx \end{pmatrix} \mathbf{B} \Rightarrow \mathbf{A}^{[-1]} \begin{pmatrix} \overset{=}{\sim} \\ \sim \\ \approx \end{pmatrix} \mathbf{B}^{[-1]} \\ \mathbf{1}^{[-1]} \overset{=}{\sim} \mathbf{1}^{\mathrm{T}} \end{gathered} \qquad (3.114)$$

For vectors, any $\mathbf{y}$ for which $\mathbf{y} \mp \left(\mathbf{x}^{[-1]}\right)^T$ will be called dimensionally dual to $\mathbf{x}$. Note that dimensional duality corresponds to a reflection about the origin in the dimensional basis space.

The following example summarizes most of the operations discussed in this section. Given

$$\mathbf{a} = \begin{bmatrix} 3 \text{ ohms} \\ 2 \text{ amps} \end{bmatrix} \tag{3.115}$$

Then:

$$\begin{gathered} \mathbf{a} \mp \begin{bmatrix} 7 \text{ ohms} \\ 0 \text{ amps} \end{bmatrix}, \ \mathbf{a} \sim \begin{bmatrix} 6 \text{ milliohms} \\ 4 \text{ kiloamps} \end{bmatrix}, \ \mathbf{a} \approx \begin{bmatrix} 4 \text{ meter-ohms} \\ 2 \text{ meter-amps} \end{bmatrix} \\ \mathbf{a}^{[-1]} \mp \begin{bmatrix} 1 \text{ ohms}^{-1} & 9 \text{ amps}^{-1} \end{bmatrix}, \ \mathbf{a} \text{ dual to } \mathbf{b} = \begin{bmatrix} 4 \text{ ohms}^{-1} \\ 3 \text{ amps}^{-1} \end{bmatrix} \end{gathered} \tag{3.116}$$

3.6.4 Dimensionally Strengthened Mathematics of Scalars

With the above notation, we can strengthen the mathematics to be valid for dimensioned quantities. The reader is referred to the first subsection of this chapter for rules on equality, multiplication, division, etc. However, an addition to these previous constraints is that the ordering relations '>', '<', and the absolute value are not distinguishable for positive or negative dimensioned quantities. For instance, we cannot say that $2 \text{ volts} > 1 \text{ volt}$, because a sign-reversal in the definition of charge (as mentioned earlier) would transform the equation to a statement $\text{-2 volts} > \text{-1 volts}$. Similarly, we cannot choose between $+a$ or $+a$ as a solution to $|a|$. To mathematically require that the physical equation remain true under any dimensional system, we must allow complete sign symmetry.

A solution to this problem is suggested by Hart, who suggests a comparison of magnitudes rather than a comparison of measurements. He defines a sign-symmetric comparison as:

$$a \succ b \text{ iff either} \left(\text{b=0 and a} \neq 0\right) \text{ or } \left(\text{b} \neq 0 \text{ and } \frac{\text{a}^2}{\text{b}^2} > 1\right) \tag{3.117}$$

Similarly:

$$a \succeq b \text{ iff either} \left(a \succ b\right) \text{ or } \left(a = b\right) \text{ or } \left(a = -b\right) \tag{3.118}$$

And the notion of inequality once again is unambiguous

3.6.5 Dimensionally Strengthened Mathematics of Vectors

For a vector consisting of an n-tuple of dimensioned scalars, we define equality as:

$$\mathbf{a} = \mathbf{b} \text{ iff } \mathbf{a}_i = \mathbf{b}_i \forall i \tag{3.119}$$

Vector addition is defined by:

$$\mathbf{a} + \mathbf{b} \;\Rightarrow \left[\mathbf{a} + \mathbf{b}\right]_i = \mathbf{a}_i + \mathbf{b}_i \text{ iff } \;\mathbf{a} \eqcirc \mathbf{b} \tag{3.120}$$

Vector scalar multiplication is defined by:

$$\mathbf{a} = c \cdot \mathbf{b} \;\text{ iff }\; \mathbf{a}_i = \mathrm{c} \cdot \mathbf{b}_i \;\forall \mathrm{i} \tag{3.121}$$

Vector outer and inner products are discussed below. Vector differentiation is defined in the standard way, by:

$$\left[\frac{\partial \mathbf{a}}{\partial b}\right]_i = \frac{\partial \mathbf{a}_i}{\partial b} \tag{3.122}$$

So $\frac{\partial \mathbf{a}}{\partial b} \eqcirc b^{-1} \cdot \mathbf{a}$. For vector differentiation:

$$\left[\frac{\partial \mathbf{a}}{\partial \mathbf{b}}\right]_{ij} = \frac{\partial \mathbf{a}_i}{\partial \mathbf{b}_j} \tag{3.123}$$

The resulting matrix has the dimensional form, $\frac{\partial \mathbf{a}}{\partial \mathbf{b}} \eqcirc \mathbf{a} \cdot \mathbf{b}^{[-1]}$.

The outer product can be defined in two ways, either via the inverse or the transpose:

$$\left[\mathbf{a} \cdot \mathbf{b}^{[-1]}\right]_{ij} \eqcirc \frac{\mathbf{a}_i}{\mathbf{b}_j} \tag{3.124}$$

$$\left[\mathbf{a} \cdot \mathbf{b}^{T}\right]_{ij} \eqcirc \mathbf{a}_i \cdot \mathbf{b}_j \tag{3.125}$$

Note that outer products involve no summation, and therefore are defined for any dimensioned vectors.

The most interesting distinction between dimensionless mathematics occurs in the case of the dot product. The conventional dot product involves a summation,

$$\mathbf{a}^T \cdot \mathbf{b} = \sum_{\mathrm{i}} \mathbf{a}_i \cdot \mathbf{b}_i \qquad (3.126)$$

Therefore, the dot product is not generally defined over vectors of arbitrary dimensions because the sum would be dimensionally incompatible. In general, dot products are used to measure magnitudes, so the desired result is dimensionless. To distinguish the dot product between the inner products, we will require the dot product to produce a dimensionless scalar, and allow inner products to produce dimensioned scalars. If we require $\mathbf{a}^T \cdot \mathbf{b}$ to be dimensionless, then each $\mathbf{a}_i \cdot \mathbf{b}_i$ should be dimensionless, which requires $\mathbf{a}_i \mathrel{\overline{\overline{\sim}}} \frac{1}{\mathbf{b}_i}$ for each i, or that $\mathbf{a} \mathrel{\overline{\overline{\sim}}} \mathbf{b}^{[-1]T}$.

Therefore the dot product is defined only when $\mathbf{a}$ and $\mathbf{b}$ are from dimensionally dual spaces. For instance, if we substitute $\mathbf{b}^{[-1]^T}$ for $\mathbf{a}$, then the dot product of 3.126 becomes:

$$\mathbf{b}^{[-1]^{T^T}} \cdot \mathbf{b} = \mathbf{b}^{[-1]} \cdot \mathbf{b} = 1 \qquad (3.127)$$

This gives a dimensionless value as expected. We also see that if two vectors are dimensionally parallel, then their dot product is dimensionless.

We invest some effort in pointing out inner product dimensional constraints because they are later found to be central to matrix mathematics. First, if the inner product $\mathbf{a}^T \cdot \mathbf{b}$ has dimensions of c, then we require that $\mathbf{a}_i \cdot \mathbf{b}_i \mathrel{\overline{\overline{\sim}}} c$. We find from direct substitution that this is equivalent to the statement:

$$\begin{aligned} \mathbf{a}^T \cdot \mathbf{b} \mathrel{\overline{\overline{\sim}}} c \ \text{ iff } \ & \mathbf{a} \mathrel{\overline{\overline{\sim}}} c \cdot \mathbf{b}^{[-1]T} \\ \text{iff } \ & \mathbf{a} \approx \mathbf{b}^{[-1]T} \end{aligned} \qquad (3.128)$$

We find later that matrix operations also yield inner products of the dimensional form $\mathbf{a}^{[-1]} \cdot \mathbf{b}$ which are defined under the following dimensional conditions on their arguments:

$$\mathbf{a}^{[-1]} \cdot \mathbf{b} \ \text{ defined iff } \ \mathbf{a} \approx \mathbf{b} \qquad (3.129)$$

Further, by transitivity:

$$\text{if } \mathbf{a}^T \cdot \mathbf{b} \text{ defined and } \mathbf{a}^T \cdot \mathbf{c} \text{ defined, then } \mathbf{b} \approx \mathbf{c} \qquad (3.130)$$

In the special case of $\mathbf{a}^T \cdot \mathbf{a}$ that appears for vector norms, we find that we require $\mathbf{a}_i^2 \eqsim \mathbf{a}_j^2 \ \forall i, j$. This places a dimensional constraint on the argument, $\mathbf{a}$, that it be a dimensionally uniform vector. In other words,

$$\mathbf{a}^T \cdot \mathbf{a} \text{ defined iff } \mathbf{a} \text{ is uniform iff } \exists \mathrm{c}, \mathbf{a} \eqsim c \cdot \mathbf{1} \qquad (3.131)$$

This is a very restrictive dimensional condition, and we find that many applications of linear algebra in physics and engineering violate these dimensional criteria. Additionally, the magnitude operator on a vector, $|\mathbf{a}|$, is defined only for dimensionless vectors. This is not as strict a problem for vector comparison, because

$$\mathbf{a}^T \cdot \mathbf{a} > \mathbf{b}^T \cdot \mathbf{b} \qquad (3.132)$$

is defined if $\mathbf{a} \eqsim \mathbf{b}$ (and both are uniform vectors). However, a problem does arise with the Holder norms:

$$\|\mathbf{a}\|_p = \left[\sum_{i=1}^{n} |a_i|^p \right]^{\frac{1}{p}} \qquad (3.133)$$

These norms are only defined for dimensionless vectors (Hart, 1995)(p. 75).

Another critical stumbling block for dimensioned vectors is the notion of orthogonality. If two vectors, $\mathbf{a}$ and $\mathbf{b}$, are from the same dimensional vector space, then $\mathbf{a} \eqsim \mathbf{b}$. If they are orthogonal, then by definition $\mathbf{a}^T \cdot \mathbf{b} = 0$ with the zero in the same dimensions as each element of the dot-product summation. From earlier arguments, the operation $\mathbf{a}^T \cdot \mathbf{b}$ is only defined when both $\mathbf{a}$ and $\mathbf{b}$ are dimensionally uniform. Therefore, the concept of orthogonality is only defined in uniform vector spaces. The notion of orthonormal vectors is still more restrictive, because normality requires $\mathbf{a}^T \cdot \mathbf{a} = 1$ and $\mathbf{a}$ uniform. However, no dimensioned value of 1 would allow a basis-independent definition of orthonormal vectors, because any dimensional terms could be changed in magnitude by dimensional conversions. Therefore, all orthonormal vectors are only defined for dimensionless vectors as their arguments.

3.6.6 Dimensionally Strengthened Mathematics of Matrices

The dimensioned operations of matrix equality, matrix addition, scalar multiplication, and differentiation with respect to a scalar all have the same dimensional requirements on the arguments as were discovered for the component-wise analysis of dimensioned vectors. However, more complex matrix operations reveal must more strict requirements on dimensioned matrices than encountered previously. Even the simple case of matrix multiplication is limited to only a small class of dimensioned matrices.

The classical definition of matrix multiplication operation generates dimensional limitations on the arguments that are more stringent than requiring that the number of rows in one matrix match the number of columns in another. The set of dimensioned matrices that satisfy the dimensional requirements of multiplication will be called **multipliable matrices,** again using Hart's notation (Hart, 1995) (p. 78). The matrix product $\mathbf{A}\cdot\mathbf{B}$ is an array of the dot products between the rows of $\mathbf{A}$ and the columns of $\mathbf{B}$. By the requirements of the dot product (discussed above), every individual row of $\mathbf{A}$ must be dimensionally parallel to each column of $\mathbf{B}$ so that the dot product, $\mathbf{a}_i \cdot \mathbf{b}_j{}^T$, is defined for every j if i is fixed, or for every i if j is fixed. For instance, if we take the first column of $\mathbf{B}$, it must be dimensionally possible to multiply it by each row of $\mathbf{A}$:

$$\begin{aligned} &\mathbf{a}_1 \cdot \mathbf{b}_1{}^T \text{ exist } \Rightarrow \mathbf{a}_1 \approx \mathbf{b}_1{}^{[-1]^T} \\ &\mathbf{a}_2 \cdot \mathbf{b}_1{}^T \text{ exist } \Rightarrow \mathbf{a}_2 \approx \mathbf{b}_1{}^{[-1]^T} \\ &\vdots \\ &\mathbf{a}_i \cdot \mathbf{b}_1{}^T \text{ exist } \Rightarrow \mathbf{a}_i \approx \mathbf{b}_1{}^{[-1]^T} \end{aligned} \qquad (3.134)$$

By equation 3.128, this implies that all the rows of the matrix $\mathbf{A}$ must be dimensionally parallel. Thus, the dimensions of every row of $\mathbf{A}$ can be written as a dimensional scalar multiplied by the first row, or simply in terms of the outer product of two dimensioned vectors:

$$\mathbf{A} \mathrel{\overline{\overline{\sim}}} \begin{bmatrix} 1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} [\text{first row of } \mathbf{A}] \mathrel{\overline{\overline{\sim}}} \mathbf{a}\cdot\mathbf{b}^T \qquad (3.135)$$

For reasons discussed later, we wish to represent the vectors as $\mathbf{b} \mathrel{\overline{\overline{\smile}}} [\text{ first row of A}]^{[-1]}$ rather than $\mathbf{b} \mathrel{\overline{\overline{\smile}}} [\text{ first row of A}]^{T}$. However, both serve to illustrate the two functions for A. We can now define the dimensional requirements on whether a matrix $\mathbf{A}$ is multipliable:

$$\begin{aligned} \mathbf{A} \text{ multipliable iff } \quad & \mathbf{A} \mathrel{\overline{\overline{\smile}}} \mathbf{a} \cdot \mathbf{b}^{[-1]} \\ \text{iff } \quad & \mathbf{A} \mathrel{\overline{\overline{\smile}}} \mathbf{a} \cdot \mathbf{b}^{T} \end{aligned} \qquad (3.136)$$

for some dimensioned vectors $\mathbf{a}$ and $\mathbf{b}$. These dimensioned vectors are called *the dimension vectors for* $\mathbf{A}$ by Hart, because their mathematical dimensions completely determine the mathematical dimensions of $\mathbf{A}$.

The requirement that the matrix product $\mathbf{A} \cdot \mathbf{B}$ is dimensionally defined is rather easily shown to be the following:

$$\begin{aligned} &\text{if } \quad \mathbf{A} \mathrel{\overline{\overline{\smile}}} \mathbf{a} \cdot \mathbf{b}^{[-1]} \text{ and } \mathbf{B} \mathrel{\overline{\overline{\smile}}} \mathbf{c} \cdot \mathbf{d}^{[-1]}, \text{ then} \\ &\mathbf{A} \cdot \mathbf{B} \text{ defined iff } \mathbf{b} \approx \mathbf{c} \end{aligned} \qquad (3.137)$$

Hart notes how this includes the traditional constraint for matrix multiplication, because $\mathbf{b} \approx \mathbf{c}$ will only hold if the number of columns of $\mathbf{A}$ match the number of rows of $\mathbf{B}$. Another way of looking at the dimensional constraints of the matrix multiplication $\mathbf{A} \cdot \mathbf{B}$ is to rewrite it as:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &\mathrel{\overline{\overline{\smile}}} \left(\mathbf{a} \cdot \mathbf{b}^{[-1]}\right) \cdot \left(\mathbf{c} \cdot \mathbf{d}^{[-1]}\right) \\ &\mathrel{\overline{\overline{\smile}}} \mathbf{a} \cdot \left(\mathbf{b}^{[-1]} \cdot \mathbf{c}\right) \cdot \mathbf{d}^{[-1]} \end{aligned} \qquad (3.138)$$

The middle term is only defined if $\mathbf{b} \approx \mathbf{c}$.

We now consider a linear equation of the form, $\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b}$ where $\mathbf{y}$, $\mathbf{x}$, and $\mathbf{b}$ are vector quantities. The previous section showed that the dimensional requirements on the addition of parameters, $\mathbf{y}$ and $\mathbf{b}$, is that $\mathbf{b} \mathrel{\overline{\overline{\smile}}} \mathbf{y}$. For $\mathbf{A}$ to be multipliable, we require $\mathbf{A} \mathrel{\overline{\overline{\smile}}} \mathbf{y} \cdot \mathbf{x}^{[-1]}$. We therefore see that pre-multiplication by $\mathbf{A}$ dimensionally transforms a vector in $\mathbf{x}$ space to a vector in $\mathbf{y}$ space. This is analogous to the mathematical meaning of a linear operator in a dimensionless mathematical viewpoint. For cascaded matrix operations, for instance $\mathbf{z} = \mathbf{B} \cdot \mathbf{y} + \mathbf{c}$, we find that a dimensional requirement of $\mathbf{c} \mathrel{\overline{\overline{\smile}}} \mathbf{z}$ and $\mathbf{B} \mathrel{\overline{\overline{\smile}}} \mathbf{z} \cdot \mathbf{y}^{[-1]}$ yields dimensionally consistent results. Specifically,

$$
\begin{aligned}
&\mathbf{z} = \mathbf{B} \cdot \mathbf{y} + \mathbf{c} = \mathbf{B} \cdot \left(\mathbf{A} \cdot \mathbf{x} + \mathbf{b} \right) + \mathbf{c} \\
&\mathbf{z} \underset{\smile}{=} \mathbf{z} \cdot \mathbf{y}^{[-1]} \cdot \left(\mathbf{y} \cdot \mathbf{x}^{[-1]} \cdot \mathbf{x} + \mathbf{y} \right) + \mathbf{z} \\
&\mathbf{z} \underset{\smile}{=} \mathbf{z}
\end{aligned}
\qquad (3.139)
$$

which is therefore dimensionally correct. Additional specific examples are given in Hart (Hart, 1995). The linear operator cases, along with the inner and outer product, illustrate the primary methods of generating dimensional constraints on the arguments to mathematical operations, and remaining discussions on dimensional mathematics will be presented in the appropriate sections of the thesis alongside application examples.

3.7 The Pi Theorem

> Dimensional analysis is a means of processing information, not providing it.
>
> \- B. S. Massey

Currently, the most important statement regarding dimensional analysis is known as the Pi Theorem and is very often referred to as the Buckingham-Pi Theorem. Although Buckingham was the first to introduce the theorem to the U.S., he was not the first to formulate the theorem (Macagno, 1971) and today the concept is generally known as the Pi Theorem. This theorem is based fundamentally on the concept of a dimensional homogeneity and the constraints of this concept on mathematical equations.

3.7.1 Variable Constraints Created by Dimensional Mappings

For any given equation, the possible grouping of the dependent and independent variables is restricted dimensionally because only a small subset of variable groupings will have appropriate dimensions. This idea is best illustrated by example.

Example: Frequency of Vibration of a Sphere of Liquid (adopted from Szirtes, (Szirtes, 1997) p. 294). An experiment is created to measure the internal mode of vibration of a liquid sphere. The liquid sphere is not under the influence of any external gravitational source, and is in equilibrium and levitating in empty space. It is assumed that the following variables are the *only* variables needed to describe the period.

Variable	Symbol	Dimension
period of vibration	T	s
sphere diameter	D	m
density	ρ	$m^{-3} \cdot kg$
gravitational constant	k	$m^3 \cdot kg \cdot s^{-2}$
surface tension	σ	$kg \cdot s^{-2}$

A formula for the period of natural vibration is desired, but the underlying dynamics are unknown. However, we know that, among the above parameters, the output variable must be measured in units of time, and therefore all terms in the equation must appear in groupings that produce this dimensional unit. The stated problem is therefore: What groupings of variables will produce the required dimensions for the measurement of the period of vibration: i.e. seconds?

Solution: By simple observation and manual manipulations of the variables, we determine that the following variables will all produce dimensions of 'seconds'.

$$\tau_1 = T$$
$$\tau_2 = \rho^{1/2} \cdot k^{1/2} \cdot \sigma^{-1}$$
$$\tau_3 = D \cdot \rho^{1/2} \cdot k^{1/6} \cdot \sigma^{-2/3}$$

Clearly, other combinations will also work. For instance:

$$\tau_4 = T^1 \cdot D^1 \cdot k^{-1/3} \cdot \sigma^{1/3}$$
$$\tau_5 = D^{3/2} \cdot \rho^{1/2} \cdot \sigma^{-1/2}$$

We will show shortly that there are an infinite number of combinations. However, the first three groupings are *independent* in the sense that *all* other possible variable combinations can be written as multiplication of these three. For instance:

$$\tau_4 = \tau_1 \cdot \tau_2^{-1} \cdot \tau_3$$
$$\tau_5 = \tau_2^{-1/2} \cdot \tau_3^{3/2}$$

Therefore, the five variables can only appear in groupings of the above three parameters if the equation is to have dimensions of seconds.

An investigation on a system with unknown governing equation is certainly benefited by the method above of minimizing the number of possible parameter groupings. For systems of many variables, a more structured approach is clearly needed than a manual, ad-hoc search. In the example above, we attempted to find variable combinations that generated 'seconds', so using the bracket notation to extract dimensions of the variables, we may write:

$$[T]^{e_1} \cdot [D]^{e_2} \cdot [\rho]^{e_3} \cdot [k]^{e_4} \cdot [\sigma]^{e_5} \underset{=}{=} \text{s} \tag{3.140}$$

Rewriting in terms of the fundamental dimensions:

$$\begin{aligned}&\left(\text{m}^0 \cdot \text{kg}^0 \cdot \text{s}^1\right)^{e_1} \cdot \left(\text{m}^1 \cdot \text{kg}^0 \cdot \text{s}^0\right)^{e_2} \cdot \left(\text{m}^{-3} \cdot \text{kg}^1 \cdot \text{s}^0\right)^{e_3} \cdot \\ &\left(\text{m}^3 \cdot \text{kg}^1 \cdot \text{s}^{-2}\right)^{e_4} \cdot \left(\text{m}^0 \cdot \text{kg}^1 \cdot \text{s}^{-2}\right)^{e_5}\end{aligned} = \left(\text{m}^0 \cdot \text{kg}^0 \cdot \text{s}^1\right) \tag{3.141}$$

Separating by dimensions:

$$\begin{aligned}0 \cdot e_1 + 1 \cdot e_2 - 3 \cdot e_3 + 3 \cdot e_4 + 0 \cdot e_5 &= 0 \leftarrow \text{m} \\ 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3 + 1 \cdot e_4 + 1 \cdot e_5 &= 0 \leftarrow \text{kg}\,. \\ 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 - 2 \cdot e_4 - 2 \cdot e_5 &= 1 \leftarrow \text{s}\end{aligned} \tag{3.142}$$

Rearranging in matrix form:

$$\begin{bmatrix} 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.143}$$

We can now define the Dimensional Matrix as follows. The definition was originally defined by Langhaar (p. 31), but was later generally adopted as standard notation, i.e. see Szirtes (Szirtes, 1997; Langhaar, 1951):

Definition - The Dimensional Matrix: Given a set of variables $V_1 \ldots V_{N_V}$, the dimensional matrix, $\mathbb{D}$, is a $N_d \times N_V$ matrix where N_d is the number of dimensions needed to express the variables, and N_V is the number of variables. The columns of $\mathbb{D}$ are formed from the basis vectors $[\![V_1]\!],[\![V_2]\!],\ldots,[\![V_{N_V}]\!]$ defined as earlier in this chapter:

$$\mathbb{D} \triangleq \left[[\![V_1]\!] \quad [\![V_2]\!] \quad \cdots \quad [\![V_n]\!] \right] \tag{3.144}$$

Notation: For a given set of variables, the dimensional matrix is not unique. Because the elements of the dimensional matrix are strongly dependent on the variable *and* ordering of the fundamental dimensions, a label is generally given to every column and row to denote exactly the meaning of each element. The example below illustrates this.

Example: For the above example, find the associated dimensional matrix, $\mathbb{D}$.

Variable	Symbol	Dimension
period of vibration	T	s
sphere diameter	D	m
Density	ρ	$\mathrm{m^{-3} \cdot kg}$
gravitational constant	k	$\mathrm{m^3 \cdot kg \cdot s^{-2}}$
surface tension	σ	$\mathrm{kg \cdot s^{-2}}$

Solution: Ordering the dimensions as $\mathrm{m \cdot kg \cdot s}$, the dimensional matrix is given in either of the two following forms:

$$\begin{array}{cc} & \begin{array}{ccccc} T & D & \rho & k & \sigma \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \mathrm{m} \rightarrow \\ \mathrm{kg} \rightarrow \\ \mathrm{s} \rightarrow \end{array} & \begin{bmatrix} 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 \end{bmatrix} = \mathbb{D} \end{array} \quad \text{or} \quad \begin{array}{c|ccccc} & T & D & \rho & k & \sigma \\ \hline \mathrm{m} & 0 & 1 & -3 & 3 & 0 \\ \mathrm{kg} & 0 & 0 & 1 & 1 & 1 \\ \mathrm{s} & 1 & 0 & 0 & -2 & -2 \end{array} = \mathbb{D}$$

Note that without the labels, the meaning of each element might be unclear, especially if the order of the variables and dimensions were not given. The second, more compact form is generally preferred.

3.7.2 Constraints on the Dimensional Matrix

The dimensional matrix just defined is constrained mathematically by some of the assumptions stated previously.

Constraint 1: *If the number of fundamental dimensions is the minimum required for a problem description, then the rank of the dimensional matrix must be equal to the number of fundamental dimensions.*

The above constraint is most evident using a proof by contradiction. First assume that the dimensional matrix does not have rank equal to the number of dimensions. For instance, if we add a temperature unit to the previous problem, then we produce a degenerate dimension.

$$\begin{array}{r|ccccc} & T & D & \rho & k & \sigma \\ \hline \text{m} & 0 & 1 & -3 & 3 & 0 \\ \text{kg} & 0 & 0 & 1 & 1 & 1 \\ \text{s} & 1 & 0 & 0 & -2 & -2 \\ {}^{\text{o}}\text{C} & 0 & 0 & 0 & 0 & 0 \end{array} = \mathbb{D}$$

The added dimension is clearly unnecessary, and the assumption of minimal dimensions is violated. For the general case where any row is linearly dependent on another row, then the corresponding dimension is not necessary, thus violating the assumptions of the constraint.

Because the number of fundamental dimensions is equal to the number of rows in the dimensional matrix, the above constraint can be restated as the following. *The rows of the dimensional matrix must be linearly independent if the number of fundamental dimensions is minimal*. This leads to the following rule:

Rule 1: *In the case where a dimensional matrix does not satisfy the above constraints, then a dimension should be eliminated to remove a row of the dimensional matrix such that the constraint is satisfied.* For instance, the temperature dimension should be removed in the previous example. Selected problems, presented later, illustrate this rule further.

Constraint 2: *For a dimensionally consistent equation written only of relevant variables, every portion of the span of the basis space of the dimensional matrix must be reachable by two or more independent parameter combinations.*

The proof of this is first illustrated by example. Let us say we add another parameter, say K, that is the only parameter that can span a particular subspace of the dimensional matrix (in this case, the subspace corresponding to Celsius degrees):

$$\begin{array}{c|cccccc} & T & D & \rho & k & \sigma & K \\ \hline \text{m} & 0 & 1 & -3 & 3 & 0 & 0 \\ \text{kg} & 0 & 0 & 1 & 1 & 1 & 0 \\ \text{s} & 1 & 0 & 0 & -2 & -2 & 0 \\ {}^{\circ}\text{C} & 0 & 0 & 0 & 0 & 0 & 1 \end{array} = \mathbb{D}_2$$

Now assume that each of the above parameters is assumed to enter into an equation. By definition of the span of the basis space, the equation can *always* be rewritten such that each term has units of ${}^{\circ}\text{C}$. However, the only parameter that can generate this dimensional unit is the parameter K. Therefore, by consistency of the equation, every term in the equation would require K to appear once and only once and only to power of one. Therefore, the equation can be equivalently rewritten without K, and therefore this parameter is not relevant.

In the general case, every reachable dimension is defined by a corresponding multiplication or division of parameters. This reachable dimension space is described directly by the span of the column vectors of the dimensional matrix. If these parameters are used in a consistent equation, then the same equation may always be rewritten in such a way that a particular, desired dimension is generated on one side of the equation. For the equation to remain consistent, a different combination of parameters (that do not cancel with the first grouping of parameters) must be capable of generating the same particular dimension on the other side of the equation by definition of consistency. Thus, every reachable point in the dimension space must be reachable by two or more independent combinations of parameters. This is expressed by the following rule:

Rule 2: *For an equation of several parameters, if only one parameter's dimension vector exists within a subspace of the dimensional matrix for the equation, then this parameter is not relevant to the equation.*

An important result of the above two rules is that the number of columns of a dimensional matrix will always be greater than the number of rows. This follows because if the number of columns is *less* than the number of rows, than one or more of the dimensions are unnecessary. Thus, the number of rows must be equal to the number of necessary dimensions. If the number of columns is *equal* to the number of rows, then each parameter's dimension must form a basis vector, and hence the parameter would not be relevant to the equation. Therefore, there must be *more* parameters than basis vectors, and thus more columns than rows.

3.7.3 Solutions to Dimensionally Constrained Problems

From Equation 3.143, we showed that an attempt to find parameter groupings with a given unit produced an equation of the form:

$$\begin{bmatrix} 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3.145)$$

The general form of the equation is:

$$\mathbb{D} \cdot \mathbf{e} = \mathbf{q} \qquad (3.146)$$

Within this equation:

- N_d is the number of fundamental dimensions, where each dimension is necessary,
- N_V is the number of relevant variables in a consistent equation, $N_V > N_d$
- $\mathbb{D}$ is a $N_d \times N_V$ matrix of rank N_d,
- $\mathbf{e}$ is a $N_V \times 1$ column vector of unknowns (which we would like to solve for), and
- $\mathbf{q}$ is a $N_d \times 1$ column vector of known (specified) dimensions,

We now consider generalized solutions to the above equation.

Because Equation 3.146 is underdetermined (it will always be by the assumptions of the problem), we must select several of the values of $\mathbf{e}$ and solve for the remainder. We therefore partition the $\mathbb{D}$ matrix into two matrices, defined as:

$$\underbrace{\mathbb{D}}_{N_d \times N_V} = \left[\underbrace{\mathbf{B}_{\mathbb{D}}}_{N_d \times (N_V - N_d)} \;\vdots\; \underbrace{\mathbf{A}_{\mathbb{D}}}_{N_d \times N_d} \right] \tag{3.147}$$

We select $\mathbf{A}_{\mathbb{D}}$ so that it is not singular, and note that this is always possible because the rows of the dimensional matrix are independent. We now rewrite Equation 3.146 as:

$$\underbrace{\left[\mathbf{B}_{\mathbb{D}} \;\vdots\; \mathbf{A}_{\mathbb{D}} \right]}_{\mathbb{D}} \cdot \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_{N_V - N_d} \\ \hline e_{N_V - N_d + 1} \\ \vdots \\ e_{N_V} \end{bmatrix}}_{\mathbf{e}} = \underbrace{\mathbf{q}}_{N_d \times 1} \tag{3.148}$$

Note that the top partition of $\mathbf{e}$ must be selected, while the bottom partition $\mathbf{e}$ can be solved for once the top portion is given. The top portion is therefore 'known', and we may rewrite as:

$$\begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_{N_V - N_d} \\ \hline e_{N_V - N_d + 1} \\ \vdots \\ e_{N_V} \end{bmatrix}}_{\mathbf{e}} = \underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_{N_V - N_d} \\ \hline q_1 \\ \vdots \\ q_{N_d} \end{bmatrix}}_{\mathbf{q}_{aug}} \tag{3.149}$$

Here the augmented q-matrix is now called $\mathbf{q}_{aug}$. The above equation has the solution:

$$\mathbf{e} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}} \end{bmatrix}^{-1} \cdot \mathbf{q}_{aug} \tag{3.150}$$

Which is simplified via the definition of the inverse of a matrix of matrices:

$$\mathbf{e} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}}^{-1} \end{bmatrix} \cdot \mathbf{q}_{aug} \tag{3.151}$$

Examples below illustrate the use of this method.

Example: Frequency of Vibration of a Sphere of Liquid (Revisited) Find the variable combinations that produce units of 'seconds' using the method of Equation 3.151.

Solution: From the previous problem of the vibrating sphere, we found the dimensional matrix as:

$$\begin{array}{c|ccccc} & T & D & \rho & k & \sigma \\ \hline \text{m} & 0 & 1 & -3 & 3 & 0 \\ \text{kg} & 0 & 0 & 1 & 1 & 1 \\ \text{s} & 1 & 0 & 0 & -2 & -2 \end{array} = \mathbb{D}$$

Which we partition as:

$$\left[\begin{array}{cc|ccc} 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 \end{array}\right] = \begin{bmatrix} \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}} \end{bmatrix}$$

So the matrix:

$$\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}}^{-1} \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1/2 & 0 & 0 & 1 & 1/2 \\ -1/2 & -1/3 & 1/3 & 1 & 1/2 \\ 1 & 1/3 & -1/3 & -1 & -1 \end{array}\right] \tag{*}$$

We now multiply this by a vector,

$$\mathbf{q}_{aug} = \left[\begin{array}{cc|ccc} e_1 & e_2 & 0 & 0 & 1 \end{array}\right]^T$$

Note that e_1 and e_2 are free variables. From linear algebra (discussed shortly), we know that there are three independent solutions. We choose reasonable values of e_1 and e_2 (left column below), multiply these by the matrix (*) to produce a solution (middle column), and then enter the solution in parameter form (right column) :

knowns $(\mathbf{q}_{aug})$	*solution* $(\mathbf{e})$	*parameters*
$[e_1 \quad e_2 \mid 0 \quad 0 \quad 1]^T$	$[e_1 \quad e_2 \mid e_3 \quad e_4 \quad e_5]^T$	$T^{e_1} D^{e_2} \rho^{e_3} k^{e_4} \sigma^{e_5}$
$[0 \quad 0 \mid 0 \quad 0 \quad 1]^T$	$[0 \quad 0 \mid 1/2 \quad 1/2 \quad -1]^T$	$T^0 D^0 \rho^{1/2} k^{1/2} \sigma^{-1}$
$[1 \quad 0 \mid 0 \quad 0 \quad 1]^T$	$[1 \quad 0 \mid 0 \quad 0 \quad 0]^T$	$T^1 D^0 \rho^0 k^0 \sigma^0$
$[0 \quad 1 \mid 0 \quad 0 \quad 1]^T$	$[0 \quad 1 \mid 1/2 \quad 1/6 \quad -2/3]^T$	$T^0 D^1 \rho^{1/2} k^{1/6} \sigma^{-2/3}$

Checking the units on the right column, this multiplication of parameters all provide the required units of seconds.

The above example illustrated that there are multiple solutions to Equation 3.151, but only a subset of them will be linearly independent. The number of linearly independent solutions depends on whether the $\mathbf{q}$ matrix is all zeros. If so, then the number of independent solutions is given by the homogenous equation, otherwise the number of solutions increases by one because of the particular solution adds one additional possible solution. Numerically, the number of solutions, N_S, is given by:

$$\begin{aligned} N_S &= N_V - N_d, & \mathbf{q} &= \mathbf{0} \\ N_S &= N_V - N_d + 1, & \mathbf{q} &\neq \mathbf{0} \end{aligned} \qquad (3.152)$$

In this equation, N_V is the number of variables and N_d is the number of necessary fundamental dimensions. See standard texts on linear algebra for details on the linear algebraic aspects of this statement (Curtis, 1997; Leon, 1998). Since each independent solution satisfies Equation 3.151, we may write:

$$\mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_{aug,1} & \mathbf{q}_{aug,2} & \cdots & \mathbf{q}_{aug,N_S} \end{bmatrix} \qquad (3.153)$$

Where $\mathbf{S}$ denotes a solution matrix. We separate the $\mathbf{q}_{aug}$ matrix back into its original components as defined in Equation 3.149:

$$\mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} & \mathbf{A}_{\mathbb{D}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} e_{11} & e_{21} & \cdots & e_{1,N_S} \\ e_{21} & e_{22} & \cdots & e_{2,N_S} \\ \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{q} & \mathbf{q} & \cdots & \mathbf{q} \end{bmatrix} \tag{3.154}$$

Denote the top matrix composed of elements, e_{ij}, as $\mathbf{E}$. Note that a requirement for the solutions to be linearly independent is that each row in $\mathbf{E}$ be linearly independent. We now write the solution in a closed form:

$$\mathbf{S} = \begin{bmatrix} \mathbf{E} \\ -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} \cdot \mathbf{E} + \mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{q} \end{bmatrix} \tag{3.155}$$

The above matrix will be denoted by the term, the **dimensional solution matrix**.

Example: Find the solution matrix, $\mathbf{S}$, for the previous problem.

Solution: From the previous problem, the $\mathbf{E}$ matrix and $\mathbf{q}$ are given as:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

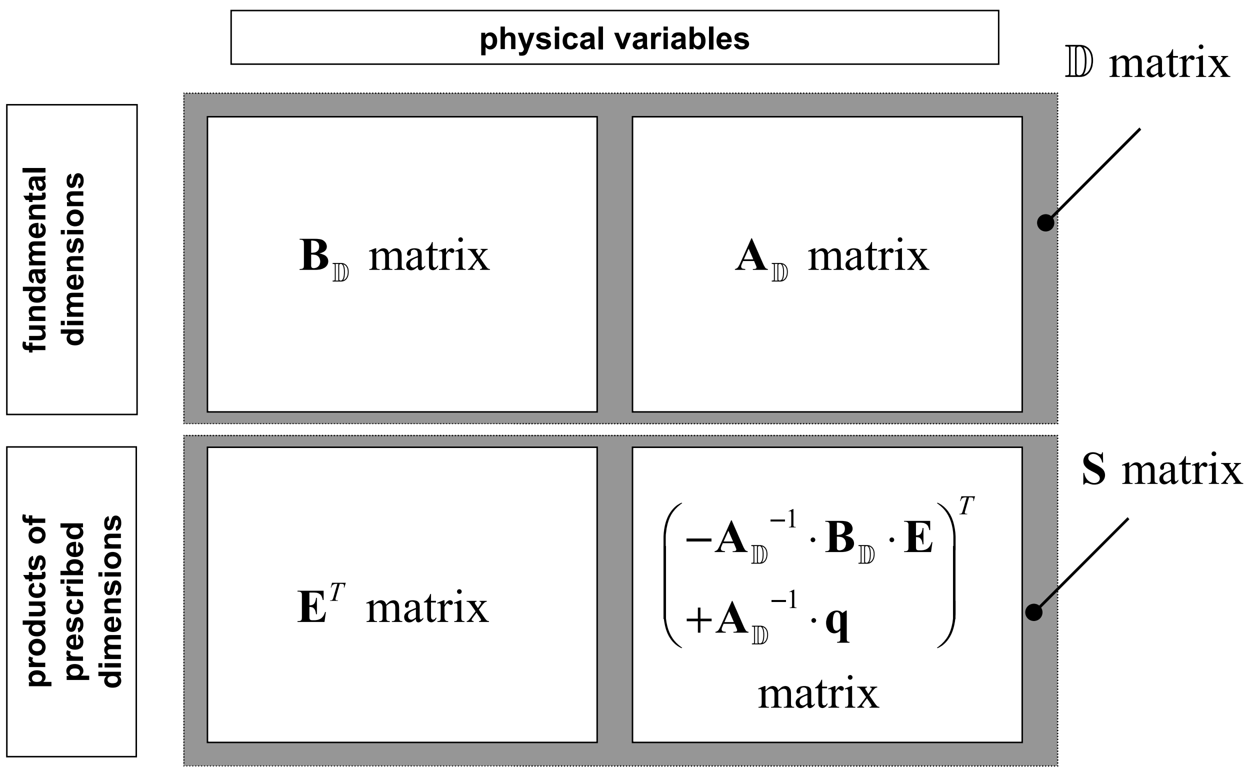
Applying the inverses defined previously, the solution matrix becomes:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/6 & 1/2 \\ 0 & -2/3 & -1 \end{bmatrix} \tag{3.156}$$

3.7.4 The Dimensional Set

By definition, the number of columns of the $\mathbf{S}$ matrix is equal to the number of solutions of Equation 3.151. The number of rows is always equal to the number of parameters, and the arrangement of the rows is such that row 1 corresponds to the variable of column 1 in the dimensional matrix, row 2 corresponds to column 2, etc.

A very compact method of presenting both the dimensional matrix and the corresponding solutions is presented in Szirtes (Szirtes, 1997) by presenting the *transpose* of the solution matrix directly under the dimensional matrix, in a new matrix that this author calls the **Dimensional Set Matrix**. The dimensional set is therefore of the form:



For example, the dimensional set for the previous example is:

	T	D	ρ	k	σ
m	0	1	−3	3	0
kg	0	0	1	1	1
s	1	0	0	−2	−2
τ_1	1	0	0	0	0
τ_2	0	1	1/2	1/6	−2/3
τ_3	0	0	1/2	1/2	−1

Note that the dimensional set compactly presents all the relevant information of the problem, including the relevant variables, the dimensional dependency, the solution method, and solutions. An interesting property of the solution matrix, $\mathbf{S}$, and the dimensional matrix, $\mathbb{D}$, is that the following is always true:

$$\mathbb{D} \cdot \mathbf{S} = \mathbf{q} \qquad (3.157)$$

This follows from straightforward multiplication.

3.7.5 Change of Dimensional Basis

We now consider the physical meaning of the transformations of Equation 3.155, which is shown again below:

$$\mathbf{S} = \begin{bmatrix} \mathbf{E} \\ -\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} \cdot \mathbf{E} + \mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{q} \end{bmatrix} \qquad (3.158)$$

This equation is best understood by re-examining the dimensional matrix of the example problem with the additional external dimensional dependency on the units of mass, length, and time shown explicitly. In Equation 3.159 below, the matrix, $\mathbf{A}_{\mathbb{D}}$, is partitioned on the right and the matrix, $\mathbf{B}_{\mathbb{D}}$, is partitioned on the left.

	T	D	ρ	k	σ	m	kg	s
m	0	1	−3	3	0	1	0	0
kg	0	0	1	1	1	0	1	0
s	1	0	0	−2	−2	0	0	1

(3.159)

It should be clear by the unitary matrix on the right-hand side that the *external* dimensions (i.e. parameters) are serving as a basis for the remaining vectors. We are free to choose any parameters we wish as dimensional basis; If we choose three *other* variables to act as dimensional basis, say ρ, k, and σ, we would multiply the above matrix through by $\mathbf{A}_{\mathbb{D}}^{-1}$ (as we have done before in the solution of the problem shown above). Thus, we obtain:

$$\begin{array}{c|ccccc:ccc} & T & D & \rho & k & \sigma & \text{m} & \text{kg} & \text{s} \\ \hline \rho & 1/2 & 0 & 1 & 0 & 0 & 0 & 1 & 1/2 \\ k & 1/2 & 1/3 & 0 & 1 & 0 & 1/3 & 1 & 1/2 \\ \sigma & -1 & -1/3 & 0 & 0 & 1 & -1/3 & -1 & -1 \end{array} \qquad (3.160)$$

Now, this would imply that we could write dimensions of T in 'units' of ρ, k, and σ. It is clear by the unity matrix under the terms ρ, k, and σ that these parameters are now serving as a new dimensional basis for the dimensions of the remaining parameters. Thus, we would measure a constant C in $\rho \cdot k \cdot \sigma$ units, and read them to someone else.

$$T = C \cdot \rho^{1/2} \cdot k^{1/2} \cdot \sigma^{-1} \qquad (3.161)$$

However, to describe the measurement, C, in a $\rho \cdot k \cdot \sigma$ unit system, we would then need to also specify our measurements of $\rho \cdot k \cdot \sigma$ which may vary. This difficulty is eliminated if we create a new dimensionless unit by dividing all terms by their corresponding dimensional powers, i.e. for the case of *T* forming a new parameter:

$$\frac{T}{\rho^{1/2} \cdot k^{1/2} \cdot \sigma^{-1}} = C \qquad (3.162)$$

Thus, we only need to report the constant, *C*. This corresponds to simply determining the product $-\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}}$ (note the minus sign) and then determining what variables (or combinations) to measure by direct multiplication with placeholders representing the appropriate dimensions, $\mathbf{E}$. Thus, the product $-\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} \cdot \mathbf{E}$ completely generates *dimensionless* measures of the remaining parameters, i.e. not included in the $\mathbf{A}_{\mathbb{D}}$ matrix. Finally, we note that the term, $\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{q}$, corresponds to a dimensional 'shift' operation in the dimensional direction of $\mathbf{q}$. Thus, if we want independent parameters with units of 'seconds', we generate the dimensionless parameters via the operation, $-\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}} \cdot \mathbf{E}$, and then shift them dimensionally in the direction of time by one time unit with the operation: $\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{q}$.

Definition (Repeating Parameters): *The parameters that are used to form a dimensional basis are called* repeating parameters *because they are repeated in every parameter set for which they serve as a dimensional basis vector.*

In summary, the previous mathematics is summarized by basis changes on dimensional subspaces. Specifically,

$$\underbrace{-\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{B}_{\mathbb{D}}}_{\substack{\text{measures parameters} \\ \text{in } \mathrm{B}_{\mathbb{D}} \text{ using new basis}}} \cdot \underbrace{\left(\mathbf{E}\right)}_{\substack{\text{specifies} \\ \text{dimensionless} \\ \text{groupings of parameters}}} + \underbrace{\mathbf{A}_{\mathbb{D}}^{-1} \cdot \mathbf{q}}_{\substack{\text{shifts dimensional} \\ \text{basis in specified} \\ \text{direction}}} \qquad (3.163)$$

The above equation arises from Equation 3.151, and the number of solutions to this equation, as already discussed, depends on the specified dimensions in the $\mathbf{q}$ vector. Clearly, the minimal parameter set occurs when the equation is reformulated as a dimensionless mapping of parameters. This is stated in the following important theorem.

3.7.6 Statement of the Pi Theorem

Theorem: *Given a function of the form* 3.164*,*

$$g\left(V_1, V_2, \ldots, V_{N_V}\right) = 0 \qquad (3.164)$$

An equivalent function of the form of 3.165 *can be written, where each pi term is dimensionless.*

$$f\left(\pi_1, \pi_2, \ldots, \pi_{N_P}\right) = 0 \qquad (3.165)$$

The number of pi variables, N_P*, in the equivalent function is always less than the dimensional form, and is given by:*

$$N_P = N_V - N_d \qquad (3.166)$$

The variable, N_V, is the number of physical parameters in the original function and N_d is the number of necessary fundamental dimensions spanned by these parameters.

By historical notation, dimensionless variables are denoted with the lower-case Greek letter, π_i, as first introduced by Buckingham. This notation is now standard; for example, see Langhaar's presentation (Langhaar, 1951) (p32). A subscript is always used in order to differentiate a pi-variable from the geometric constant, pi, of numerical value 3.14159 etc.

3.7.7 Proof of the Pi Theorem

This theorem is attributed to Buckingham (Buckingham, 1914) (p. 345) who presented it in 1914, along with its proof based on Euler's Homogenous Function Theorem. A history by Macagno is given on the subject (Macagno, 1971), that notes that Vaschy first stated this theorem (without proof) in 1892 (in French). A discussion of this paper at the Royal Academy was brought to the attention of Buckingham, who conceded later that Vachy's work motivated his work. Unfortunately, the theorem is almost universally, but incorrectly, credited to Buckingham.

The following authors have all provided forms of proofs for the Pi Theorem:

- E. Buckingham
- K. Brenkert,
- P. Bridgman, who presents the proof using Euler's homogenous equation method in Chapter 4 of his book, (Bridgman, 1943). The proof is fairly complete and straightforward, but there are a few notational mistakes that may initially confuse the reader. Bridgman makes the important point that equations that are not homogenous violate the assumptions of the proof.
- H. Langhaar, who presents the proof on the basis of the invariance of dimensional mappings in Chapter 4 of his book (Langhaar, 1951). Langhaar introduces Euler's form on the last section of the last chapter, but does not use it to prove the theorem directly.
- G. Birkhoff
- W. Durand
- S. Drobot
- L Brand

- L. Sedov
- Duncan (1953) presents a very complete form of the proof based on Euler's homogenous theorem method (plus examples of meaning of homogenous equations) (Duncan, 1953).
- Isaacson and Isaacson: (1975) presents a fairly complete form of the proof (Isaacson and Isaacson, 1975) (p 29-32) as a form of Euler's homogenous equation method.

Historically, only the more recent theorems have examined dimensional span issues as presented in this thesis. The proof of this approach was derived directly from the discussion above. Specifically, the parameters utilized in the formation of the $\mathbf{A}_{\mathbb{D}}$ matrix are fully constrained dimensionally. Thus, only the remaining parameters can enter the equation independently. The fact that the set is minimal (i.e. there is no smaller parameter set representation based on dimensional arguments) arises from the result that the maximal number of parameters that can be used to span the dimensional space of the problem always generate dimensionless products.

3.7.8 Examples

Some examples are given to illustrate the basic concepts. A notable result of dimensional analysis is that, in many cases, the governing equation of a system can be determined solely from dimensional considerations.

Example: Deflection of a Cantilever Upon a Concentrated Lateral Load (Adopted from Szirtes, (Szirtes, 1997), p 141,143,146,149)

A cantilever of uniform cross section is loaded with a point force, and the deformation is assumed to be strictly in the lateral direction. The figure below shows a diagram of the arrangement.

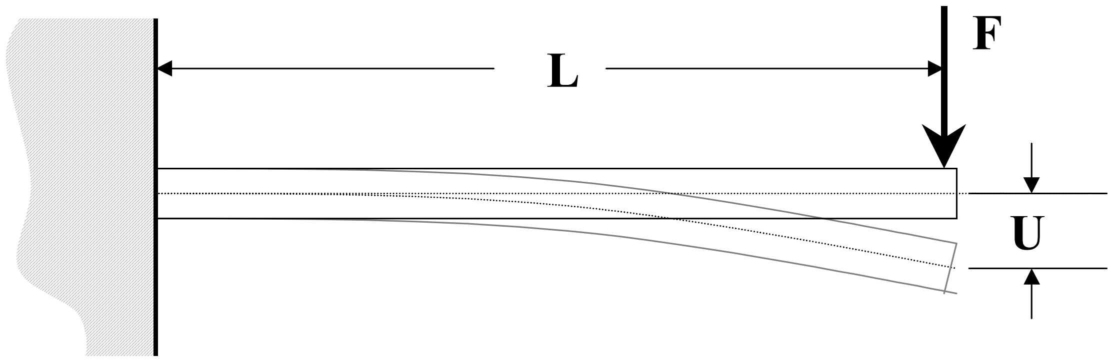


Figure 3.4: Cantilever deflection

The relevant variables, their symbols and dimensions are listed as:

Variable	Symbol	Dimension
lateral deflection	U	m
lateral load	F	N
length	L	m
Young's modulus	E	$m^{-2} \cdot N$
diameter of x-section	D	m

(1) Determine an independent set of variable groupings that are dimensionless.

(2) Show that, if SI units are used, the number of dimensions is not minimal.

Solution: Part (1): From the Pi Theorem, there are:

$$\begin{aligned} N_P &= N_V - N_d \\ &= 5 - 2 \\ &= 3 \end{aligned} \qquad (3.167)$$

independent dimensionless variable groupings. For the dimensional set matrix, we choose the **E** matrix as the 3×3 identity matrix, and solve for the remainder of the dimensional set. The dimensional set is therefore given as (Note that the **q** vector is zero, and will be assumed to be zero hereafter unless otherwise stated):

$$
\begin{array}{c|ccc|cc}
 & U & F & L & E & D \\
\hline
\text{m} & 1 & 0 & 1 & -2 & 1 \\
\text{N} & 0 & 1 & 0 & 1 & 0 \\
\hline
\pi_1 & 1 & 0 & 0 & 0 & -1 \\
\pi_2 & 0 & 1 & 0 & -1 & -2 \\
\pi_3 & 0 & 0 & 1 & 0 & -1
\end{array}
\qquad (3.168)
$$

The dimensionless groupings can be read directly from the bottom three rows as:

$$
\pi_1 = U \cdot D^{-1}, \pi_2 = F \cdot E^{-1} \cdot D^{-2}, \pi_3 = L \cdot D^{-1} \qquad (3.169)
$$

Part (2): In SI units, the dimensions of the variables are as follows:

Variable	Symbol	Dimension
lateral deflection	U	m
lateral load	F	$\mathrm{m \cdot kg \cdot s^{-2}}$
length	L	m
Young's modulus	E	$\mathrm{m^{-1} \cdot kg \cdot s^{-2}}$
diameter of x-section	D	m

The Pi Theorem now predicts $N_P = N_V - N_d = 5 - 3 = 2$ independent, dimensionless variable groupings. For the Dimensional Set Matrix, we choose the **E** matrix as the 2×2 identity matrix, and solve for the remainder of the dimensional set. The dimensional set is therefore given as:

$$
\begin{array}{c|cc|ccc}
 & U & F & L & E & D \\
\hline
\text{m} & 1 & 1 & 1 & -1 & 1 \\
\text{kg} & 0 & 1 & 0 & 1 & 0 \\
\text{s} & 0 & -2 & 0 & -2 & 0 \\
\hline
\pi_1 & 1 & 0 & ? & ? & ? \\
\pi_2 & 0 & 1 & ? & ? & ?
\end{array}
\qquad (3.170)
$$

Note that the $\mathbf{A}_{\mathbb{D}}$ matrix is singular! The dimensional rows are linearly dependent, since the "second" row is twice the "kg" row. We must choose one of these two to delete, so arbitrarily we remove seconds. The Pi Theorem again predicts $N_P = N_V - N_d = 5 - 2 = 3$ independent, dimensionless variable groupings, and the dimensional set becomes.

	U	F	L	E	D
m	1	1	1	−1	1
kg	0	1	0	1	0
π_1	1	0	0	0	−1
π_2	0	1	0	−1	−2
π_3	0	0	1	0	−1

(3.171)

The pi groups become:

$$\pi_1 = U \cdot D^{-1}, \pi_2 = F \cdot E^{-1} \cdot D^{-2}, \pi_3 = L \cdot D^{-1} \qquad (3.172)$$

which are exactly as determined previously.

The above example illustrates that the pi-groups of a system are independent of the dimensional system chosen. Another important result of the Pi Theorem is that, if the number of dimensionless variables is equal to one, then the function must be a constant. This follows from the fact that if an equation of one variable is equal to a constant for arbitrary values of the variable, then the variable must be constant. The following example illustrates a historically important application of dimensional analysis, one that generated a significant amount of controversy:

Example: Propagation of the Blast Wave Front in an Atomic Explosion (Adapted from Szirtes (Szirtes, 1997), p. 154. This example actually occurred in the mid-1950's, and the history is given dramatically in both Barenblatt's and McMahon's books (McMahon and Bonner, 1983; Barenblatt, 1996).

Given a release of a very large amount of energy at a point location, a blast wave of pressure is created that approximates a bubble. Inside the expanding bubble, there is a large amount of pressure buildup while outside the bubble there is atmospheric pressure. The only relevant variables are:

Variable	Symbol	Dimension
radius of wave front	R	m
Time	t	s
initial air density	ρ_0	$m^{-3} \cdot kg$
released energy	Q	$m^2 \cdot kg \cdot s^{-2}$

Find the pi values for the associated equation.

Solution: Part (1): From the Pi Theorem, there are $N_P = N_V - N_d = 4 - 3 = 1$ independent dimensionless variable groupings. For the dimensional set matrix, we choose the **E** matrix as unity, and solve for the remainder of the dimensional set. The dimensional set is therefore given as:

$$\begin{array}{c|c:ccc} & R & t & \rho_0 & Q \\ \hline m & 1 & 0 & -3 & 2 \\ kg & 0 & 0 & 1 & 1 \\ s & 0 & 1 & 0 & -2 \\ \hdashline \pi_1 & 1 & -2/5 & 1/5 & -1/5 \end{array} \qquad (3.173)$$

The dimensionless groupings can be read directly from the bottom row as:

$$\pi_1 = R \cdot t^{-2/5} \cdot \rho_0^{1/5} \cdot Q^{-1/5} \qquad (3.174)$$

Since there is only one pi-variable, we know that this is a constant!

$$constant = R \cdot t^{-2/5} \cdot \rho_0^{1/5} \cdot Q^{-1/5}$$

This relationship has been confirmed from measurements of blast waves from atomic blasts, shown in the figure below (figure is from McMahon (McMahon and Bonner, 1983)):

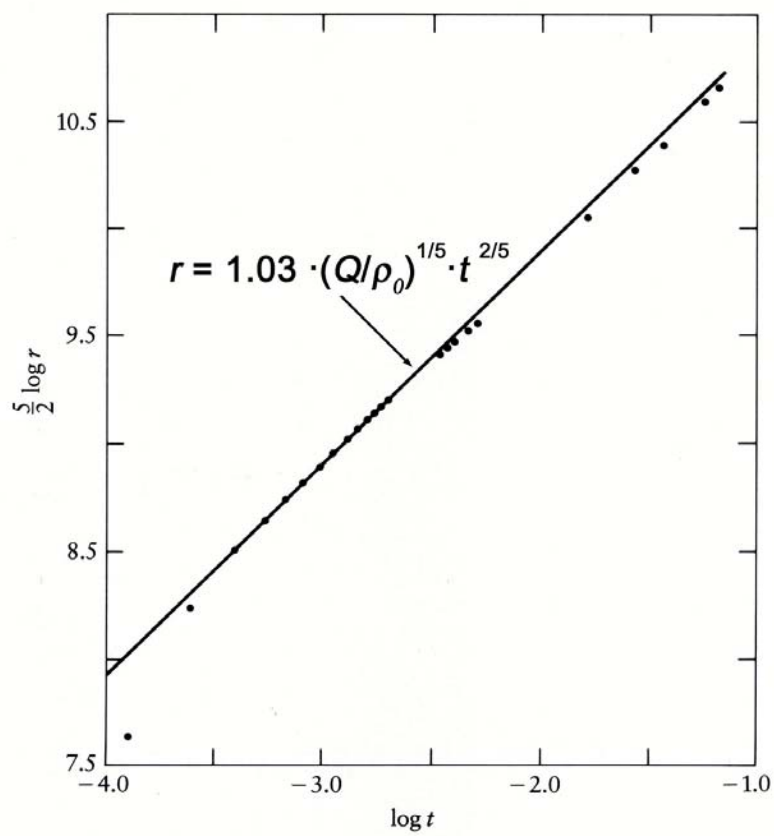


Figure 3.5: Data confirming single pi term relationship for atomic blasts

Example: Radiation Pressure on Satellites (Adapted from Szirtes (Szirtes, 1997), p. 177.)

The radiation pressure by the Sun on space vehicles affects their orbits significantly over time. The relevant variables are:

Variable	Symbol	Dimension
radiation pressure	p	$m^{-1} \cdot kg \cdot s^{-2}$
radiating power of the	Q	$m^{2} \cdot kg \cdot s^{-3}$
speed of light	c	$m \cdot s^{-1}$
Sun-Earth distance	R	m

Find the pi values for the associated equation.

Solution: Part (1): From the Pi Theorem, there are $N_P = N_V - N_d = 4 - 3 = 1$ independent dimensionless variable groupings. For the dimensional set matrix, we choose the $\mathbf{E}$ matrix as unity, and solve for the remainder of the dimensional set. The dimensional set is therefore given as:

	p	Q	c	R
m	−1	2	1	1
kg	1	1	0	0
s	−2	−3	−1	0
π_1	1	−1	1	3

(3.175)

The dimensionless groupings can be read directly from the bottom row, and because there is just one dimension, we know it is a constant as:

$$\pi_1 = p \cdot Q^{-1} \cdot c \cdot R^3 = C \qquad (3.176)$$

Since there is only one pi-variable, we know that this is a constant! In theory, this constant is easily derivable, and is equal to $1/4\pi$. If one substitutes values corresponding to Earth, the pressure is $p = 4.5 \times 10^{-6}$ Pa $= 6.6 \times 10^{-10}$ psi. This is exceedingly small, but over a period of years this constant pressure can cause significant deviation to a satellite orbit.

Controls Application Example: Planar Motion of a Vehicle at High Speed

We wish to investigate the pi-parameters governing vehicle motion at highway speeds. For purposes of analysis, we assume that suspension, pitch, and roll effects are negligible, and that the motion of a vehicle at high speed on the highway is primarily limited to motion along the plane of the road. This motion is governed by Newton's laws. Using the standard model for vehicle dynamics as a guideline (See Chapter 2, and Appendices C and D), the following parameters are found to be significant:

Variable	Symbol	Dimension
length from front axle to C.G.	a	m
length from rear axle to C.G.	b	m
front tire forces produced per unit slip	$C_{\alpha f}$	$\mathrm{m \cdot kg \cdot s^{-2}}$
rear tire forces produced per unit slip	$C_{\alpha r}$	$\mathrm{m \cdot kg \cdot s^{-2}}$
z-axis moment of inertia	I_z	$\mathrm{m^2 \cdot kg}$
mass of the vehicle	m	kg
length of the vehicle	L	m
speed of the vehicle	U	$\mathrm{m \cdot s^{-1}}$

Find the pi values for the associated equation.

Solution: Part (1): By the Pi Theorem, there are $N_P = N_V - N_d = 8 - 3 = 5$ independent dimensionless variable groupings. Thus, there are three basic unit dimensions: mass, length, and time, abbreviated M, Le, and T. It is somewhat intuitive to choose the vehicle mass (*m*), vehicle length (*L*), and the 'vehicle length travel time' (*L*/*U*) to represent fundamental units in the three dimension spaces. The remaining five unused parameters can be re-cast as dimensionless Π groups by appropriate division or multiplication of *m*, *L* and *U*.

For completeness of the presentation of dimensional analysis, it must be mentioned that one method to determine the parameters is a manual approach. This approach is very tedious, but is still the most common application of the Pi Theorem. As an example of this method, the pi-groups formed for the vehicle dynamics problem are derived first manually and then using the matrix formulation developed earlier.

To explain the manual method, a dimensional equation is formed in powers of the chosen repeating parameters. For instance:

$$C_{\alpha f} \cdot m^a \cdot U^b \cdot L^c = \left[\frac{M * Le}{T^2}\right] \cdot [M]^a \cdot \left[\frac{Le}{T}\right]^b \cdot [Le]^c = [M * Le * T]^0 . \qquad (3.177)$$

Equating the powers, three equations are obtained:

$$\begin{aligned} &mass && 1+a=0 \\ &time && -2-b=0 \\ &length && 1+b+c=0. \end{aligned} \qquad (3.178)$$

Solving the equations gives a = -1, b = -2, and c = 1. Hence, one pi-group is $C_{\alpha f} \cdot L / m \cdot U^2$.

Solving for a second pi-group related to the moment of inertia, I_z:

$$I_z \cdot m^a \cdot U^b \cdot L^c = \left[M * Le^2\right] \cdot [M]^a \cdot \left[\frac{Le}{T}\right]^b \cdot [Le]^c = [M * Le * T]^0 \ . \qquad (3.179)$$

Equating the powers, three equations are obtained:

$$\begin{aligned} &mass && 1+a=0 \\ &time && -b=0 \\ &length && 2+b+c=0. \end{aligned} \qquad (3.180)$$

Solving the equations gives a = -1, b = 0, and c = -2. Therefore, this pi-group is $I_z / m \cdot L^2$.

A more methodical approach is offered by the matrix method developed previously. Because we want m, L, U as repeating parameters, we ensure that they are in the right-most column of the dimensional matrix. For the dimensional set, we choose the **E** matrix as unity, and solve for the remainder of the dimensional set. The dimensional set is therefore given as:

	a	b	$C_{\alpha f}$	$C_{\alpha f}$	I_z	m	L	U
m	1	1	1	1	2	0	1	1
kg	0	0	1	1	1	1	0	0
s	0	0	−2	−2	0	0	0	−1
π_1	1	0	0	0	0	0	−1	0
π_2	0	1	0	0	0	0	−1	0
π_3	0	0	1	0	0	−1	1	−2
π_4	0	0	0	1	0	−1	1	−2
π_5	0	0	0	0	1	−1	−2	0

(3.181)

The dimensionless groupings can be read directly from the bottom rows:

$$\pi_1 = \frac{a}{L}, \pi_2 = \frac{b}{L}, \pi_3 = \frac{C_{\alpha f} \cdot L}{mU^2}, \pi_4 = \frac{C_{\alpha r} \cdot L}{mU^2}, \pi_5 = \frac{I_z}{mL^2} \qquad (3.182)$$

Note that each of the pi-groups contains physical meaning about the system. Pi parameters 1, 2, and 5 all deal with geometric similarity. Two identically constructed vehicles with only change in length scales will automatically have a match between these parameters. Note also that the additional pi-groups related to the tire radius, *r*, and wheel track, *d,* are neglected because the test vehicle construction maintained constant length scaling, and this scaling is represented fully by parameters 1 and 2. Indeed, a later discussion shows that parameter 2 is governed fully by parameter 1 because both represent the same requirement of geometric similarity.

The two remaining parameters, π_3 and π_4, can be shown to be the *Vehicle Froude Number*, and it is shown later (Chapter 6) that any mechanical system dominated by Newton's Laws will require similarity of this parameter. The Froude number defines the ratio between inertial acceleration and gravitational accelerations, and matching of the Froude number requires that the following ratio remain constant:

$$\pi_{Froude} = \frac{V}{\sqrt{L \cdot g}} \tag{3.183}$$

In this equation, V is some characteristic velocity, L is some characteristic length, and g is the force of gravity. The vehicle parameters can be simplified to Equation 3.184 where the variable, k, is a constant.

$$\pi_4 = k \cdot \pi_3 \tag{3.184}$$

With the additional assumption that the cornering forces are approximately proportional to the weight on the vehicle tires:

$$\begin{aligned} &C_{\alpha f} \propto m \cdot g \\ &\pi_3 \propto \frac{m \cdot g \cdot L}{mU^2} = \frac{g \cdot L}{U^2} = \left(\frac{1}{\pi_{Froude}} \right)^2 \end{aligned} \tag{3.185}$$

Therefore, matching the Froude number will infer necessarily provide matching of the parameters, π_3 and π_4. The Froude number was originally investigated by Froude in 1850, and stands as one of the first and primary Newtonian pi-parameters ever discovered. It governs similarity between any dynamic system with inertial accelerations and whose external forces are a function of gravity. The dynamics of wave motion, sailing vessels, ice boats, and rubber-tyred vehicles all involve a form of the Froude number (see Taylor, (Taylor, 1974), p. 75), and a dimensionless formulation of the Lagrange equations presented later shows a clear dependence on the Froude number of a system.

3.8 Dimensional Constraints on System Representations

3.8.1 State-Space Equations: Dimensional and Dimensionless Forms

The standard state-space representations of a linear dynamic system are:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\cdot\mathbf{x}(t)+\mathbf{B}\cdot\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\cdot\mathbf{x}(t)+\mathbf{D}\cdot\mathbf{u}(t)\end{aligned} \tag{3.187}$$

From the notation established earlier in this chapter, these matrix representations and associated vector operations impose dimensional requirements on the **A**, **B**, **C**, and **D** matrices of:

$$\begin{aligned}\mathbf{A} &\mathrel{\overset{}{\underset{\smile}{=}}} \dot{\mathbf{x}}\cdot\mathbf{x}^{[-1]}\\ &\mathrel{\underset{\smile}{=}} \frac{1}{t}\mathbf{x}\cdot\mathbf{x}^{[-1]},\\ \mathbf{B} &\mathrel{\underset{\smile}{=}} \dot{\mathbf{x}}\cdot\mathbf{u}^{[-1]}\\ &\mathrel{\underset{\smile}{=}} \frac{1}{t}\mathbf{x}\cdot\mathbf{u}^{[-1]},\\ \mathbf{C} &\mathrel{\underset{\smile}{=}} \mathbf{y}\cdot\mathbf{x}^{[-1]},\\ \mathbf{D} &\mathrel{\underset{\smile}{=}} \mathbf{y}\cdot\mathbf{u}^{[-1]}\end{aligned} \tag{3.188}$$

Because any matrix of the dimensional form $\mathbf{x}\cdot\mathbf{x}^{[-1]}$ will have a dimensionless diagonal term, the formation of a system $\mathbf{A}$ matrix of the dimensional form $\frac{1}{t}\mathbf{x}\cdot\mathbf{x}^{[-1]}$ implies that the elements on the diagonal must have dimensions of $\frac{1}{t}$. Therefore, all eigenvalues (or any other diagonalization of $\mathbf{A}$) will also share these dimensions, as expected.

System transforms are often needed to convert a dimensioned system representation to a dimensionless representation are obtained by a state substitution that normalizes each state, **x**, to a dimensionless form $\bar{\mathbf{x}}$, each input $\mathbf{u}$ to a dimensionless form $\bar{\mathbf{u}}$, each output $\mathbf{y}$ to a dimensionless form $\bar{\mathbf{y}}$, and each time t to a dimensionless form $\bar{t}$.

$$\mathbf{x} = \mathbf{N}_x \cdot \bar{\mathbf{x}} \qquad \mathbf{u} = \mathbf{N}_u \cdot \bar{\mathbf{u}} \qquad \mathbf{y} = \mathbf{N}_y \cdot \bar{\mathbf{y}} \qquad t = \mathbf{N}_t \cdot \bar{t} \tag{3.189}$$

The dimensions on $\mathbf{N}_x$, $\mathbf{N}_u$, $\mathbf{N}_y$, and $\mathbf{N}_t$ must satisfy the dimensional relationships:

$$\mathbf{N}_x \mp \mathbf{x} \cdot \mathbf{1}^{\sim} \qquad \mathbf{N}_u \mp \mathbf{u} \cdot \mathbf{1}^{\sim} \qquad \mathbf{N}_y \mp \mathbf{y} \cdot \mathbf{1}^{\sim} \qquad \mathbf{N}_t \mp t \cdot \mathbf{1}^{\sim} \tag{3.190}$$

In these equations, $\mathbf{1}^{\sim}$ is a vector of *dimensionless* elements of appropriate dimension. The simplest representation of each of the dimensionally normalizing matrices is created by using a diagonal matrix where the diagonal terms are composed of required repeating parameters to cancel the required dimensions.

The state-space equations can be rewritten in the states $\bar{\mathbf{x}}$ and time unit $\bar{t}$ by first noting that the derivative term must be dimensionless as well:

$$\frac{d\mathbf{x}}{dt} = \frac{d\left(\mathbf{M} \cdot \bar{\mathbf{x}}\right)}{dt} = \frac{1}{\mathbf{S}} \cdot \mathbf{M} \cdot \frac{d\bar{\mathbf{x}}}{dt} \tag{3.191}$$

Substitution of Equations 3.189 into 3.187 gives:

$$\begin{aligned} \mathbf{N}_t^{-1} \cdot \mathbf{N}_x \cdot \frac{d\bar{\mathbf{x}}}{d\bar{t}} &= \mathbf{A} \cdot \mathbf{N}_x \cdot \bar{\mathbf{x}} + \mathbf{B} \cdot \mathbf{N}_u \cdot \bar{\mathbf{u}} \\ \mathbf{N}_y \cdot \bar{\mathbf{y}} &= \mathbf{C} \cdot \mathbf{N}_x \cdot \bar{\mathbf{x}} + \mathbf{D} \cdot \mathbf{N}_u \cdot \bar{\mathbf{u}} \end{aligned} \tag{3.192}$$

Rewriting:

$$\begin{aligned} \frac{d\bar{\mathbf{x}}}{d\bar{t}} &= \mathbf{N}_t \cdot \mathbf{N}_x^{-1} \cdot \mathbf{A} \cdot \mathbf{N}_x \cdot \bar{\mathbf{x}} + \mathbf{N}_t \cdot \mathbf{N}_x^{-1} \cdot \mathbf{B} \cdot \mathbf{N}_u \cdot \bar{\mathbf{u}} \\ \bar{\mathbf{y}} &= \mathbf{N}_y^{-1} \cdot \mathbf{C} \cdot \mathbf{N}_x \cdot \bar{\mathbf{x}} + \mathbf{N}_y^{-1} \cdot \mathbf{D} \cdot \mathbf{N}_u \cdot \bar{\mathbf{u}} \end{aligned} \tag{3.193}$$

An equivalent dimensionless system can be found for any linear state-space representation of the form:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}} \cdot \bar{\mathbf{x}} + \bar{\mathbf{B}} \cdot \bar{\mathbf{u}} \\ \bar{\mathbf{y}} &= \bar{\mathbf{C}} \cdot \bar{\mathbf{x}} + \bar{\mathbf{D}} \cdot \bar{\mathbf{u}} \end{aligned} \tag{3.194}$$

The dimensional and dimensionless system matrices are related by following transforms, obtained by inspection:

$$\begin{aligned} \bar{\mathbf{A}} &= \mathbf{N}_t \cdot \mathbf{N}_x^{-1} \cdot \mathbf{A} \cdot \mathbf{N}_x \ , \quad & \bar{\mathbf{B}} &= \mathbf{N}_t \cdot \mathbf{N}_x^{-1} \cdot \mathbf{B} \cdot \mathbf{N}_u \\ \bar{\mathbf{C}} &= \mathbf{N}_y^{-1} \cdot \mathbf{C} \cdot \mathbf{N}_x \ , \quad & \bar{\mathbf{D}} &= \mathbf{N}_y^{-1} \cdot \mathbf{D} \cdot \mathbf{N}_u \end{aligned} \qquad (3.195)$$

The above equations can be used to directly transform any linear dimensional state-space system equation into the dimensionless counterpart, and vice versa. An alternative approach is to solve for the dimensionless state-space matrices by performing normalization of each dimensional state-space matrix element-by-element. This alternative, but lengthy procedure, parallels the manual approach presented earlier, and is presented in earlier work by the author (Brennan, 1999).

Situations commonly arise where the dimensions of the states might be arbitrary and only the input/output behavior of the system is of concern. For example, the dimensions of internal states of a controller are usually not constrained; only the input/output dimensional mapping is needed to transform dimensionless controllers to dimensional forms. For conversions such as these where the internal state dimensions are not important, the most obvious choice is to choose a dimensionless state vector and attach known dimensions to the inputs and outputs of the system as needed. Specifically, choose:

$$\mathbf{x} \underset{\smile}{=} \mathbf{1} \qquad \mathbf{N}_x \underset{\smile}{=} \mathbf{1} \cdot \mathbf{1}^{\sim} \qquad \mathbf{N}_y \underset{\smile}{=} \dim(\mathit{output}) \quad \mathbf{N}_u \underset{\smile}{=} \dim(\mathit{input}) \qquad (3.196)$$

The above transformations are illustrated below in the dimensional transformation of planar vehicle dynamics.

Example: Dimensionless form of Vehicle Dynamics

To demonstrate the use of the Pi Theorem, we use the linear vehicle model presented earlier as an example. From earlier sections in this chapter, the governing parameters and pi-values were found to be:

	a	b	$C_{\alpha f}$	$C_{\alpha f}$	I_z	m	L	U
m	1	1	1	1	2	0	1	1
kg	0	0	1	1	1	1	0	0
s	0	0	−2	−2	0	0	0	−1
π_1	1	0	0	0	0	0	−1	0
π_2	0	1	0	0	0	0	−1	0
π_3	0	0	1	0	0	−1	1	−2
π_4	0	0	0	1	0	−1	1	−2
π_5	0	0	0	0	1	−1	−2	0

(3.197)

The dimensionless groupings can be read directly from the bottom rows:

$$\pi_1 = \frac{a}{L}, \pi_2 = \frac{b}{L}, \pi_3 = \frac{C_{\alpha f} \cdot L}{mU^2}, \pi_4 = \frac{C_{\alpha r} \cdot L}{mU^2}, \pi_5 = \frac{I_z}{mL^2} \qquad (3.198)$$

Note that angles such as the steer angle, yaw angle, and slip angle are unitless and thus form their own Π groups.

The Buckingham Pi theorem states that if two dynamic systems are described by the same differential equations, the solution to these equations will be the same if the Π parameters are the same. This becomes clear in the dimensional analysis of the governing differential equations, which are presented later in this thesis. Examining the system characteristic equation given in Chapter 2, we note that the s term has units of (sec^{-1}). If we form a dimensionless characteristic equation (with starred terms indicating dimensionless quantities), it becomes:

$$s^{*2} + \left((\pi_3 + \pi_4) + \frac{1}{\pi_5}\left(\pi_1^2 \pi_3 + \pi_2^2 \pi_4\right) \right) s^* + \frac{1}{\pi_5}\left(\pi_3 \pi_4 - \pi_1 \pi_3 + \pi_2 \pi_4\right) = 0 \ . \qquad (3.199)$$

Clearly, if the Π groups match between two systems governed by the bicycle model, the normalized pole locations will be the same. Thus, a high degree of dynamic similitude between two systems will necessarily require that the two systems have nearly the same characteristic equation. However, it is clear that the two pole locations in the characteristic equation do not uniquely determine the five pi groups, hence **similar pole locations do not guarantee dynamic similitude!** Similarity in pole locations is a necessary, but not sufficient, condition for dynamic similitude.

In matrix form, the vehicle system is normalized using the repeating parameters to normalize the state vector. The time normalization becomes:

$$t = \frac{L}{U} \cdot t^* \Rightarrow \mathbf{S} = \frac{L}{U} \tag{3.200}$$

The state-space equations can be rewritten in the states **x*** using a state renormalization matrix:

$$\mathbf{M} = diag\begin{bmatrix} L & U & 1 & \frac{U}{L} \end{bmatrix} \tag{3.201}$$

Through the transforms given previously, or through substitution of the pi groups directly, one would find that the non-dimensional state-space form is given symbolically as:

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\pi_3 - \pi_4 & \pi_3 + \pi_4 & -\pi_1 \cdot \pi_3 + \pi_2 \cdot \pi_4 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-\pi_1 \cdot \pi_3 + \pi_2 \cdot \pi_4}{\pi_5} & \frac{\pi_1 \cdot \pi_3 - \pi_2 \cdot \pi_4}{\pi_5} & -\frac{\pi_1^2 \cdot \pi_3 + \pi_2^2 \cdot \pi_4}{\pi_5} \end{bmatrix}, \tag{3.202}$$

$$\mathbf{B}^* = \begin{bmatrix} 0 & 0 \\ \pi_3 & \pi_4 \\ 0 & 0 \\ \frac{\pi_1 \cdot \pi_3}{\pi_5} & -\frac{\pi_2 \cdot \pi_4}{\pi_5} \end{bmatrix} \tag{3.203}$$

Which results in the non-dimensional form of the transfer function:

$$\frac{\overline{y}(s^*)^*}{\delta_f(s^*)} = \frac{1}{s^{*2}} \frac{\pi_3 \cdot s^{*2} + \frac{\pi_2 \cdot \pi_3 \cdot \pi_4}{\pi_5} \cdot s^* + \frac{\pi_3 \cdot \pi_4}{\pi_5}}{s^{*2} + \left[\pi_3 + \pi_4 + \frac{\pi_1^2 \cdot \pi_3 + \pi_2^2 \cdot \pi_4}{\pi_5}\right] \cdot s^* + \frac{\pi_3 \cdot \pi_4}{\pi_5} - \frac{\pi_1 \cdot \pi_3 - \pi_2 \cdot \pi_4}{\pi_5}} \quad (3.204)$$

Note that $s \Rightarrow s^*$ because s has dimensions of $1/t$ and must also be normalized.

3.8.2 System versus Signal Normalization

Many authors studying control theory have already developed heuristic normalization techniques that partially approximate the dimensional analysis approach just presented. For instance, before determining signal norms, Skogestad and Postlethwaite (Skogestad and Postlethwaite, 2000) suggest adding additional scaling transforms to limit the largest control effort, tracking error, and reference input to all have unit max norms. To do this, one uses a simple variable transformation that implicitly forms a dimensionless version of each of the signals of interest.

To illustrate these implicit approaches, we again present the dimensional system representation in the equations below:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{u} \\ \mathbf{y} &= \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u} \end{aligned} \quad (3.205)$$

Skogestad and Postlethwaite suggest a signal-norm approach where one infers beforehand a maximum allowable bound on particular signals of interest. This is very similar to the normalization method suggested by Khalil (Khalil, 1996). The measured signals are then rescaled using the maximum bounds as elements in a diagonal matrix, as shown in the equations below.

$$\begin{aligned} \mathbf{u} &= \mathbf{D}_u \cdot \mathbf{u}_n \\ \mathbf{e} &= \mathbf{D}_e \cdot \mathbf{e}_n \\ \mathbf{r} &= \mathbf{D}_r \cdot \mathbf{r}_n \end{aligned} \quad (3.206)$$

The subscript n refers to *normalized*, meaning that $\mathbf{u}_n$, $\mathbf{e}_n$, and $\mathbf{r}_n$ are all constrained to be within $[-1,1]$ for each channel. Mathematically, the scaling matrices are defined as:

$$
\begin{aligned}
\mathbf{D}_u &= diag(\max(\mathbf{u}_i)) \\
\mathbf{D}_e &= diag(\max(\mathbf{e}_i)) \\
\mathbf{D}_e &= diag(\max(\mathbf{r}_i))
\end{aligned}
\qquad (3.207)
$$

Note that $\mathbf{y}$ must be dimensionally similar to $\mathbf{e}$, since:

$$
\mathbf{e} = \mathbf{r} - \mathbf{y} \qquad (3.208)
$$

We may now write:

$$
\mathbf{D}_e \cdot \mathbf{e}_n = \mathbf{r} - \mathbf{y} \Rightarrow \mathbf{e}_n = \mathbf{D}_e^{-1} \cdot \mathbf{r} - \mathbf{D}_e^{-1} \cdot \mathbf{y} \qquad (3.209)
$$

Since

$$
\mathbf{e}_n = \mathbf{r}_n - \mathbf{y}_n \Rightarrow \mathbf{y}_n = \mathbf{D}_e^{-1} \cdot \mathbf{y} \qquad (3.210)
$$

We finally obtain a state transformation that allows inputs of $\mathbf{u}_n$, $\mathbf{r}_n$, and measured output $\mathbf{y}_n$. This is given by:

$$
\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \mathbf{A}_n \cdot \mathbf{x} + \mathbf{B}_n \cdot \mathbf{u}_n \\
\mathbf{y}_n &= \mathbf{C}_n \cdot \mathbf{x} + \mathbf{D}_n \cdot \mathbf{u}_n
\end{aligned}
\qquad (3.211)
$$

with

$$
\begin{aligned}
\mathbf{A}_n &= \mathbf{A} \\
\mathbf{B}_n &= \mathbf{B} \cdot \mathbf{D}_u \\
\mathbf{C}_n &= \mathbf{D}_e^{-1} \cdot \mathbf{C} \\
\mathbf{D}_n &= \mathbf{D}_e^{-1} \cdot \mathbf{D} \cdot \mathbf{D}_u
\end{aligned}
\qquad (3.212)
$$

Note that the reference input, $\mathbf{r} = \mathbf{D}_r \cdot \mathbf{r}_n$, must be scaled before entering the system equations. The scaled states are unaffected. The above representation is not unique; one might have ‘moved’ the scaling gains entirely to the B or C matrices or portioned the scaling factors arbitrarily between the two matrices.

Another example is the numerical normalization required for proper matrix conditioning with the goal of minimizing errors due to finite-word-length approximations to real numbers.

The common scaling procedure (for instance, see the 'Scaling' sections of MATLAB help files or the LINPACK user guides) is to select:

$$\mathbf{x} = \mathbf{N}_x \cdot \mathbf{x}_n \qquad \mathbf{u} = \mathbf{N}_u \cdot \mathbf{u}_n \qquad \mathbf{y} = \mathbf{N}_y \cdot \mathbf{y}_n \tag{3.213}$$

The resulting normalized representation is given by:

$$\begin{aligned} \dot{\mathbf{x}}_n &= \mathbf{A}_n \cdot \mathbf{x}_n + \mathbf{B}_n \cdot \mathbf{u}_n \\ \mathbf{y}_n &= \mathbf{C}_n \cdot \mathbf{x}_n + \mathbf{D}_n \cdot \mathbf{u}_n \end{aligned} \tag{3.214}$$

where

$$\begin{aligned} \mathbf{A}_n &= \mathbf{N}_x^{-1} \cdot \mathbf{A} \cdot \mathbf{N}_x \,, \quad & \mathbf{B}_n &= \mathbf{N}_x^{-1} \cdot \mathbf{B} \cdot \mathbf{N}_u \\ \mathbf{C}_n &= \mathbf{N}_y^{-1} \cdot \mathbf{C} \cdot \mathbf{N}_x \,, \quad & \mathbf{D}_n &= \mathbf{N}_y^{-1} \cdot \mathbf{D} \cdot \mathbf{N}_u \end{aligned} \tag{3.215}$$

The suggestion for the normalization matrices is to choose the maximum range of each input, state, and output variables.

The result of the normalization based on the dimensionless transforms of Equation 3.195, the normalization based on signals suggested by Postlethwaite and Khalil in Equation 3.212, and the normalization based on numerical balanced issues of Equation 3.215 are all very similar. However, some distinguishing qualities must be mentioned. The signal-based and numeric-based approaches avoid time normalization, which prevents insight into problems with time-varying parameters (discussed later). The signal-based approach yields a fully-dimensional A matrix, the numerics-based approach yields a uniform A matrix, and the dimensionless approach yields a dimensionless A matrix. Additionally, the signal-based and numerics-based approaches choose the normalization variables randomly, yet the Pi-Theorem just presented suggests that careful selection of repeating parameters within these matrices may significantly reduce the number of parameters in the governing equations. Additional motivation for dimensionless representations will be presented later.

3.9 Contributions of This Chapter

The primary contributions of this chapter are as follows, numbered by relation to corresponding sections of the chapter:

(1) Demonstrate that there is a rich history of dimensional analysis, and that the key contributors to this field include many of the greatest scientists, engineers, and mathematicians of humanity..
(2) Introduce basic notions of physical dimensions and their use in basic measurements.
(3) Present the basic unit systems in use today and dispel the notion that any one system may be ‘superior’ to another.
(4) Demonstrate how to convert between different dimensioning systems (i.e. unit systems) and discuss how the use of a unit system is generally based on an assumption of the Absolute Significance of Relative Magnitude, and that some ‘measurement’ systems violate this assumption.
(5) Argue that the mathematical use of dimensioned quantities requires an implicit, structured, and carefully constrained set of mathematical operations that are dependent on the dimensions of the arguments.
(6) Argue that mathematical operations on dimensioned quantities is best represented by operations on an ordered pair consisting of a real term and a vector quantity of rational numbers. The use of such mathematical orderings:
    a. Imposes a sign-symmetry on all physical descriptions.
    b. Shows that dimensioned mathematics is not closed under addition
    c. Constrains arguments to most mathematical functions
    d. Is extendable to specialized forms of vectors and matrices
(7) Argue that dimensional constraints of the vector form above limit allowable forms of physical equations. Specifically,
    a. There are a limited number of possible variable combinations for a given problem that can satisfy unit constraints necessary in the equation solution.
    b. The possible variable combinations from which a solution set must exist can be generalized to a set of linear dimension-vector equations.
    c. These dimension-vector equations are always under-determined.
    d. A reparameterization to dimensionless parameter forms is obtainable by a partial-solution to the dimension-vector equations and always reduces the number of parameters in an equation description

(8) Introduce basic dimensionless representations of system equations. These forms:

a. Are derivable either by direct variable parameterizations or by simple state-substitutions in state-space forms combined with a temporal renormalization.

b. Directly generalize the results of numerical balancing and normalization methods generally used in numerical analysis techniques.

c. Demonstrate (by example) that similar pole locations do not guarantee dynamic similitude (This topic is of such importance that it is discussed in great detail in following chapter).

3.10 References

1. Barenblatt, Grigory Isaakovich. Scaling, Self-Similarity, and Intermediate Asymptotics. Cambridge: Cambridge University Press, 1996.

2. Brennan, Sean . "Modeling and Control Issues Associated With Scaled Vehicles." Masters Thesis. University of Illinois at Urbana-Champaign, 1999.

3. Bridgman, P. W. Dimensional Analysis. 3rd printing, revised ed. New Haven: Yale University Press, 1943.

4. Buckingham, E. "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations." Physical Review 4, 2nd series (1914): 345-76.

5. Curtis, Charles W. Linear Algebra, An Introductory Approach. 1974. New York: Springer, 1997.

6. Duncan, W. J. Physical Similarity and Dimensional Analysis: An Elementary Treatise. London: Edward Arnold & Co., 1953.

7. Ellis, Brian. Basic Concepts of Measurement. 1st ed. Cambridge: University Press, 1966.

8. Focken, C. M. Dimensional Methods and Their Applications. 1st ed.: Edward Arnold & Co., 1953.

9. Hart, George W. Multidimensional Analysis, Algebras and Systems for Science and Engineering. New York: Springer-Verlag, 1995.

10. Huntley, H. E. Dimensional Analysis. 1st ed. London: MacDonald & Co., 1952.

11. Isaacson, E. de St. Q., and M. de St. Q. Isaacson. Dimensional Methods in Engineering and Physics. 1st ed. New York: John Wiley & Sons, 1975.

12. Khalil, Hassan K. Nonlinear Systems. Second ed. Upper Saddle River, NJ 07458: Prentice Hall, 1996.

13. Langhaar, Henry. Dimensional Analysis and Theory of Models. 1st ed. New York: Wiley and Sons, 1951.

14. Leon, Steven J. Linear Algebra With Applications. 1980. Fifth ed. Upper Saddle River, New Jersey 07458 : Prentice Hall, 1998.

15. Macagno, Enzo O. "Historico-Critical Review of Dimensional Analysis." Journal of the Franklin Institute 292.6, Dec. (1971): 391-402.

16. McMahon, Thomas A., and John Tyler Bonner. On Size and Life. New York: Scientific American Books, Inc., 1983.

17. O'Rahilly, A. Measures and Units Vol. 1, Ch. 3. Physico-Chemical Methods: Methuen, 1939.

18. Skogestad, Sigurd, and Ian Postlethwaite. Multivariable Feedback Control, Analysis and Design. 1996. Baffins Lane, Chichester England: John Wiley & Sons Ltd., 2000.

19. Szirtes, Thomas. Applied Dimensional Analysis and Modeling. 1st ed. New York: McGraw-Hill, 1997.

20. Taylor, Edward S. Dimensional Analysis for Engineers. Oxford: Clarendon Press, 1974.

Chapter 4

Sensitivity Analysis and System Equivalency

The notion of parametric sensitivity is generally a secondary consideration in the study of dynamic behavior, with a primary consideration given to nominal stability and behavior. If one considers a generalized space where time-varying problems lie along one axis, state-varying problems lie along another axis, and parameter-varying problems along another, one might obtain a generalized figure as seen below. Specifically, we may partition separate areas of study by the nature of equations generally considered in the field.

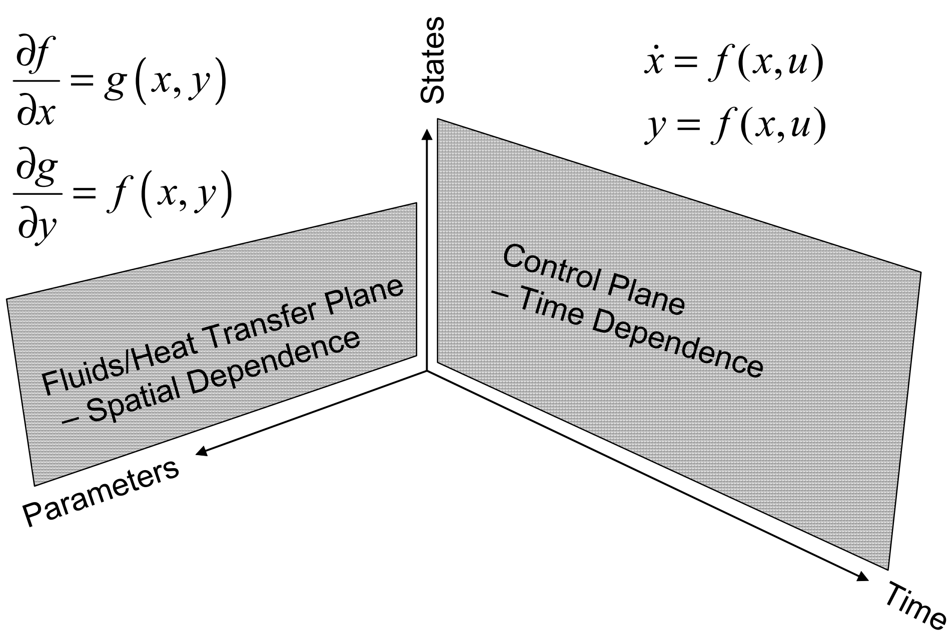


Figure 4.1: Generalization of domains of study

This chapter analyzes the use of dimensional analysis and the statements of system equivalency that have been previously applied to spatial-parametric problems to compensate for variations in space (length). From the spatial-parametric problems of fluid dynamics and heat transfer, it is well known that dimensional analysis can collapse the parameter dependence of system descriptions in the spatial-parametric domain. In these fields, a statement of dynamic similitude implies that one system is experimentally equivalent to another. However, the mathematical meaning of this statement and the system properties that it implies remain unclear in the standard system notion of control theory.

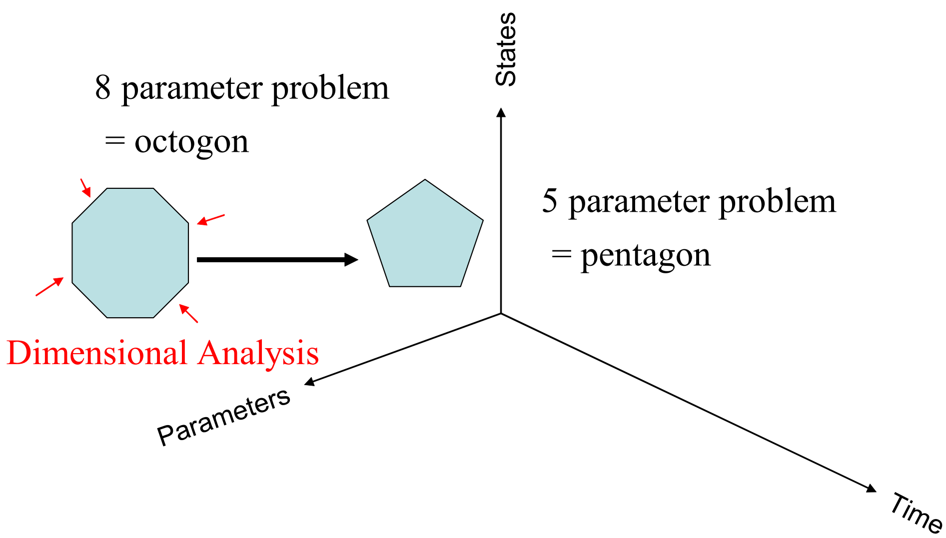


Figure 4.2: Common use of dimensional analysis in spatial-parametric domain

While the design of a controller for a nominal system generally ignores parameter variation, the intent and purpose of a feedback controller design is insensitivity to parameter variations, unmodeled dynamics, and external disturbances. Consider the requirement of insensitivity to parameter variations; this analysis can be imagined as examining variations of the trajectory (time domain) or Bode plot (frequency domain) for small projections in the parameter domain as shown in Figure 4.3. Rather than a simple state-time problem structure, the analysis space enters the state-parameter-time domain. In this sense, the problem begins to enter the consideration of the spatial-parametric-temporal domain and therefore methods useful in the spatial-parametric domain might apply.

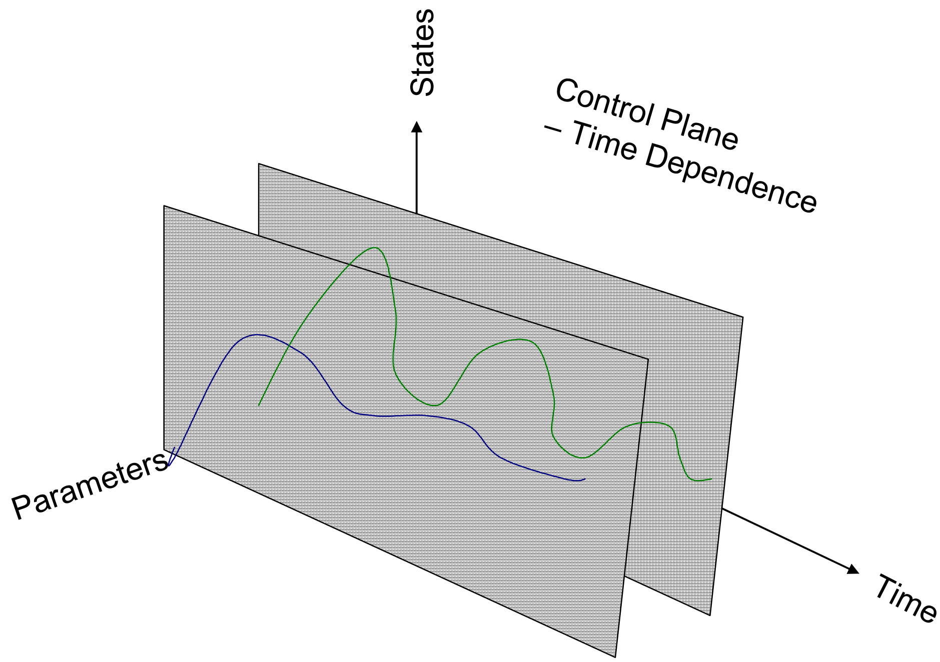


Figure 4.3: How controls analysis overlaps spatial-parametric domain

In the analysis of dimensional techniques on control problems, it will be found that the notion of dynamic similarity in the spatial-parametric sense usually applied for fluids or heat-transfer problems extends directly to the notion of system sensitivity in a spatial-temporal sense of control theory.

To summarize this chapter, a discussion is first presented demonstrating sensitivity analysis is a unifying approach for addressing model uncertainty, unmodeled dynamics, and external disturbances. The implication is that the parametric analysis presented throughout this chapter and thesis is readily extendible to notions of disturbance reduction, model sensitivity, and robustness from both a signal and systems-perturbation perspective. Next, a historical overview of the field of sensitivity is first presented and some fundamentals of sensitivity theory are introduced. Euler's Homogenous Function Theorem (EHF Theorem) is then presented and consequences to sensitivity analysis are discussed. The intent of the presentation of the EHF Theorem is to prove the degree and nature of parameter interdependency on a physical equation due to the constraints of dimensional homogeneity. General statements are then made on sensitivity invariance and associated invariant sensitivity subspaces. Ties to empirical historical studies are discussed. The notion of coupled sensitivity equations is then extended to reparameterization of the governing system model, and this analysis is shown to lead to the Pi Theorem result presented in Chapter 2. The Pi Theorem defines a notion of equivalence of dimensionless model representations based on equivalence of pi-parameters, and this new notion of system equivalence is discussed where two systems are dynamically similar if they share three properties: (1) nominal equivalence in the dimensionless domain in the sense of Perkins and Cruz, discussed later, (2) equivalent model sensitivity which implies equivalent model substructure and parameter dependence, and (3) equivalent invariant subspaces. Distinctions between this notion of equivalence and classical control notions, such as equivalent pole-zero locations, are discussed.

Based on the notion of system equivalence, a discussion is given to general model classes consisting of nominally dynamically-similar systems. Within each class, a discussion is given on the optimization with respect an exogenous cost function, and such optimization will imply additional subspace invariance in sensitivity functions. The high level of sensitivity coupling that must exist in all physical systems is shown to generalize during experimentation to power-law relationships between system parameters. General examples of optimized system classes are then given in the fields of biology and mechanical systems that exhibit such relationships.

4.1 Sensitivity Equivalence of Parameter Variations and Disturbance Inputs

The two most important goals of automatic control are (1) to reduce errors due to external disturbances and (2) reduce errors due to change in system parameters. This chapter primarily focuses only on the second topic: parameter variations and the effect of parameter variation on system behavior. While this focus may seem to address only *half* the intent of control, the following discussion will show that one can conceptually study disturbances as instead fictitious variations in the nominal plant model, or vice versa. Both goals can be unified in a single study of system sensitivity with respect to either concept.

To illustrate the dual nature of disturbance rejection and parameter insensitivity, consider the following control loop:

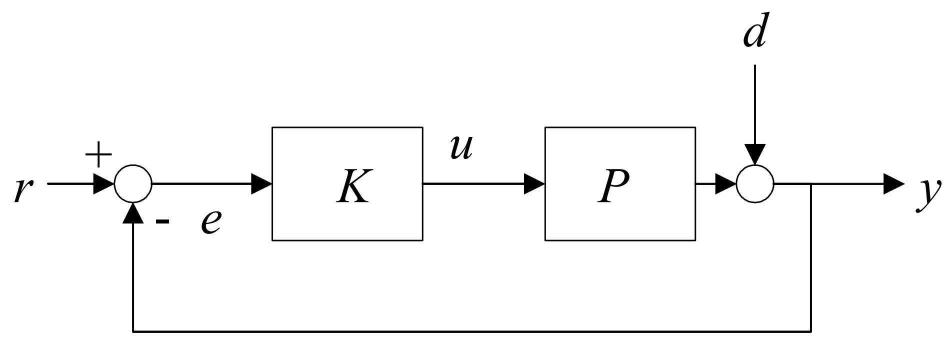


Figure 4.4: A simple 1 DOF control loop

The plant is assumed to have a nominal value of P_0. The nominal closed-loop transfer function is therefore given as:

$$G_0 = \frac{P_0 K}{1 + P_0 K} \tag{4.1}$$

If we examine the relative change in the closed-loop for a given, relative change in the plant, we obtain the classical sensitivity measure of Bode (discussed in more detail later):

$$\begin{aligned} S_P^G &= \frac{\partial G}{\partial P} \cdot \frac{P}{G}\bigg|_{P_0} \\ &= \frac{K \cdot (1 + P_0 \cdot K) - P_0 \cdot K^2}{(1 + P_0 \cdot K)^2} \cdot \frac{P_0}{G_0} \\ &= \frac{K}{(1 + P_0 \cdot K)^2} \cdot P_0 \cdot \frac{(1 + P_0 \cdot K)}{P_0 \cdot K} \\ &= \frac{1}{1 + P_0 \cdot K} \end{aligned} \tag{4.2}$$

The same result is obtained if one uses the comparison approach of Horowitz (Frank, 1978) (p. 250) that does not assume an infinitesimal perturbation, or the comparison method of Perkins and Cruz, again both of which are discussed shortly. For a given control input, the ratio of change in output to the change in plant is therefore given by:

$$\frac{\Delta Y / Y_0}{\Delta P / P_0} = \frac{1}{1 + P_0 \cdot K} \tag{4.3}$$

Now consider the Laplace transform of the error due to inputs to the system, d and r. Mathematically,

$$\begin{aligned} E &= R - Y \\ &= R - \frac{P_0 K}{1 + P_0 \cdot K} R - \frac{1}{1 + P_0 \cdot K} D \qquad (4.4) \\ E &= \frac{1}{1 + P_0 \cdot K} \cdot (R - D) \end{aligned}$$

We find that:

$$\frac{E}{R - D} = \frac{1}{1 + P_0 \cdot K} \qquad (4.5)$$

This is identical to the result of Equation 4.2. This demonstrates that the sensitivity of the system with respect to plant variations is identical to that of the sensitivity to relative reference signals, $R - D$. Therefore, if a control loop with one degree-of-freedom is designed to achieve a certain behavior with respect to reference tracking or disturbance response, then it possesses the same behavior with respect to relative changes in the plant due to internal model variations. In a more practical sense, given a measured output signal from a single-degree-of-freedom plant that exhibits a certain characteristic in the error response (like steady-state offset when given a step input), then it is impossible on this information alone to determine whether the error is due to internal model variations or external disturbances.

This duality between disturbance rejection and internal plant variations is well-known, but it is generally employed to show that disturbances can represent plant perturbations, or more specifically, exogenous disturbances can model internal plant variations and uncertainties. This is the primary method by which modern robust control techniques analyze model uncertainty within a nominal plant model, and this technique is utilized in later chapters of the thesis. However, the generality of the parametric sensitivity approach presented in this chapter relies on the opposite notion, that exogenous parametric or other general plant perturbations can represent disturbance inputs. For this reason, a dimensionless sensitivity analysis to disturbances is not presented, and methods for dealing with dimensionless signals and systems are presented in later chapters that extend readily to disturbance inputs.

4.2 Background Material

> Sensitivity considerations have provided a fundamental motivation for the use of feedback and are largely responsible for its development into what is called modern control theory, implying the principles of optimization and adaptation.
>
> - Paul M. Frank

The traditional notion of a system generally implied in literature can be divided into two notions: the structure of a system and the system parameters. The system structure is generally characterized by (Frank, 1978):

- The order of the differential or difference equation
- The linearity of nonlinearity of the model
- The rationality or irrationality and relative degree of a transfer function

The system parameters generally consist of:

- Initial conditions
- Time-varying or time-invariant coefficients
- Natural frequencies, pulse frequencies
- Sampling periods, sampling instants
- Pulse widths or magnitudes
- Length of time delays

An easy way to distinguish the two definitions is offered by dimensional analysis: system structure determines the dimension of a pi-space (the number of pi-parameters needed), while the parameters of a system characterize the range-space of the pi-parameters (the numerical values of the pi-parameters).

4.2.1 History of Sensitivity Analysis

A history of Sensitivity Analysis is provided in Frank (Frank, 1978) and more recently in Eslami (Eslami, 1994), and reviews of the subject can be found by Kokotovic (Kokotovic, 1986; Kokotovic and Rutman, 1965), Ngo (Ngo, 1971), among many others. Despite the tremendous amount of available literature on the subject, major contributors to the field deserve mention because of their historical contributions directly to this thesis. The fundamental importance of sensitivity in the design of control systems was established by Bode, who introduced a proper sensitivity definition on the basis of the frequency domain. Many of the ideas presented in this section are direct extensions of Bode's original work from the mid-1940's (Bode, 1945).

An irony in the field of automatic control theory is that in the decade following Bode's work, the study of sensitivity is not usually discussed in academic texts on the subject. An exception is the work of Horowitz (Frank, 1978) who developed the concept in the frequency domain (Horowitz, 1963).

The development of the digital computer and a state-space approach during the 1960's gave rise to a renewed interest in the subject, and the most notable developments are from the work of Kokotovic and colleagues, Perkins and Cruz, and Kreindler. Kokotovic and Perkins have historical associations with the author's institution of study, namely the University of Illinois at Urbana-Champaign. This has provided a subtle but direct influence on this author and this work. In a more direct manner, it was Perkins who directly taught the author the entirety of classical feedback control over the course of three classes and a year-and-a-half time period. It was Perkins wonderful lectures at the University of Illinois that introduced the author to the subject of sensitivity theory, and discussions with Perkins on references to study have provided very solid and invaluable introduction to the subject.

4.2.2 Basic Concepts of Sensitivity Theory

The role of sensitivity analysis is to determine the change in system behavior due to parameter variations. We define the parameters of a system as a vector,

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \cdots & p_{Np} \end{bmatrix}^T \tag{4.6}$$

It is assumed that the mathematical model of the system (which may be nonlinear) can be given by the vector differential equation:

$$\frac{d\mathbf{x}}{dt} = f\left(,\mathbf{p},t,\mathbf{u}\right) \tag{4.7}$$

where $\mathbf{x}$ represents the state vector of the system with the initial state $\mathbf{x}\left(t_0\right) = \mathbf{x}^0$ and $\mathbf{u}$ represents the input vector. This equation relates the state vector $\mathbf{x}$ to the parameter vector, $\mathbf{p}$. We wish to investigate the change in the nominal equation, $\mathbf{x}_0$, as the nominal parameter, $\mathbf{p}_0$, is perturbed, i.e. $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$. The mapping between a subspace of perturbed parameters, $R_{\mathbf{p}}$, and the subspace of perturbed states, $R_{\mathbf{x}}$, is shown in Figure 4.5:

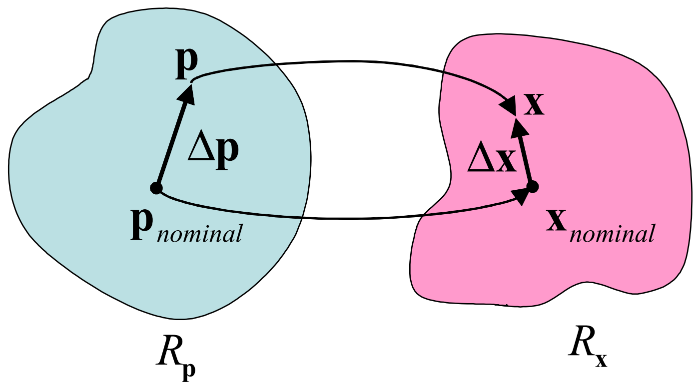


Figure 4.5: Mapping of parameter variation to state variation

4.2.3 The Miller-Murray Classification of Parameter Variation

To analyze the concept of sensitivity from a general view, parameter sensitivity is generally classified by the different methods required for its treatment (Frank, 1978; Eslami, 1994). For continuously acting systems, the classification by Miller and Murray is most often used (Miller and Murray, 1953). The following three classifications of parameter variations are given by definition:

Definition (α-errors): *Parameter variations around a nominal value* α_0 *that do not affect the order of the mathematical model are called* **α-errors** (or α-variations).

A necessary condition for errors to be α-errors is that $\alpha_0 \neq 0$. Common sources of α-errors would be identification inaccuracies, manufacturing tolerances, or changes in environmental or operating conditions. The existence theorem of differential equations states that the output of the system is continuous in α if the corresponding differential equation is continuous in α. Therefore, a system function can be considered an analytic function of α_0 (Frank, 1978). Parameters affected by α-errors will be called α-parameters.

Definition (β-errors): *Parameter variations of the initial conditions away from their nominal value* β_0 *are called* **β-errors** (or β-variations).

A typical source of β-errors is measurement inaccuracies, inexact adjustments, or the presence of noise or other disturbances. The output of a continuous system is always continuous in the initial conditions, so the system function is also an analytic function of β_0 (Frank, 1978). Parameters affected by β-errors will be called β-parameters.

Definition (λ-errors): *Parameter variations that change the order of the mathematical model when they are varied away from their nominal value* λ_0 *are called* **λ-errors** (or λ-variations or more commonly singular perturbations).

Sources of λ-errors include idealizations of mathematical model that neglect certain dynamics, or the erroneous deletion of the order of a system model. The dependence of the output of a system on λ-errors is not analytic, and therefore the sensitivity function is not analytic for these perturbations. The term 'singular perturbation' has been adopted to indicate the singularity of the sensitivity function for these types of variations at a nominal value of $\lambda_0 = 0$. Parameters of this type are called λ-parameters.

In summary, α-errors affect the parameters of the model, β-errors affect the initial (or boundary) conditions of the model solution, and λ-errors affect the order of the mathematical model. This thesis is organized somewhat around the Miller-Murray classification system. Chapter 4 (this chapter) deals primarily with α-errors, Chapter 5 addresses λ-errors in the context of system modeling, and Chapter 7 discusses how β-error sensitivity becomes important in the control and modeling of nonlinear systems.

4.2.4 Common Definitions and Measures of Sensitivity

A general means of separating sensitivity measures is to classify them by the domain in which the control design is implemented. Thus, three primary domains arise:

1. Time-domain classifications
2. Frequency-domain classifications
3. Performance-index domain classifications

The domain of primary usage is the frequency domain for simplicity of model analysis and controller design. The primary drawback of time-domain approaches is that they generally require closed-form solutions for specific inputs, so a generalizable approach is difficult to obtain. The primary drawback of performance-index approaches is that it may be impossible to compare the sensitivity of open-loop and closed-loop performance indices, even if the two systems are nominally equivalent. This problem is called the Pagurek-Witsenhausen paradox after the two authors who first discovered this issue. For these reasons, the following discussion of sensitivity measures will focus primarily on the frequency domain, especially on the traditional sensitivity measures of Bode, Horowitz, and the comparison method of Perkins and Cruz.

We introduce the concept of the sensitivity function $\mathbf{S}$ which maps to first order a parameter perturbation to the state perturbation:

$$\Delta\mathbf{x} \approx \mathbf{S}\left(\mathbf{p}_0\right)\cdot\Delta\mathbf{p} \tag{4.8}$$

Note that the sensitivity function will exist under certain continuity conditions not discussed in this thesis. See the books by Frank and Eslami (Frank, 1978; Eslami, 1994) for details. In the case that the parameter perturbation is a column vector, then the sensitivity function becomes a row vector. In the case that the governing equation is a vector-based equation, the sensitivity function becomes a matrix. As an example, the well-known Jacobian matrix is a sensitivity matrix related to the Absolute Sensitivity Function, defined below (Frank, 1978) (p14).

There are many notions of system sensitivity, and thus a general sensitivity analysis must distinguish between the different concepts. There are three basic notions of sensitivity:

(1) The sensitivity function of Bode, or Bode sensitivity, based on infinitesimal perturbations
(2) The sensitivity function of Horowitz, based on small (delta) perturbations,
(3) The sensitivity function of Perkins and Cruz, or comparison sensitivity, based on the comparison of nominally equivalent open-loop and closed-loop systems

The primary focus of the sensitivity analysis presented in this Chapter is on the traditional Bode sensitivity. However, many (and possibly most) of the results generalize to the Horowitz and comparison definitions of sensitivity. Frank discusses the similarity of the methods extensively in his book on the topic (Frank, 1978), and it is clear from this reference that much of the analysis presented in this chapter extends well to these alternate definitions of sensitivity.

A discussion of Bode sensitivity begins with the definition of the Absolute Sensitivity Function:

$$\mathbf{S}_p^f = \left.\frac{\partial f}{\partial p}\right|_{p_0} \tag{4.9}$$

The subscript denotes that partial-derivative is evaluated at the nominal parameter values, if not otherwise specified, it may be assumed that the parameter above is a α-parameter by the Miller-Murray classification. It will be implicitly assumed that the above partial-derivatives exist, and therefore the governing equations are smoothly dependent on the parameters or subsystems over which the partial derivative is examined. The smoothness requirement is often stated in existence theorems for the solutions to differential equations. However, in the specific case of switched (hybrid) systems and discrete systems, this condition can be relaxed by a redefinition of the sensitivity on each side of the switching discontinuity; again, see Frank's book for details.

A formal analysis into system sensitivity from an automatic control perspective was first considered by Bode, who defined the sensitivity as the change in parameter with respect to the change in the system (Bode, 1945). Today, the Bode sensitivity function is actually the reciprocal ratio he defined in his original analysis. The modern form, defined below, is based on what is known as the relative sensitivity function, defined as:

$$\overline{\mathbf{S}}_p^f = \left.\frac{\partial f / f}{\partial p / p}\right|_{p_0} = \left.\frac{p_0}{f_0}\frac{\partial f}{\partial p}\right|_{p_0} \tag{4.10}$$

In general, the bar-notation is dropped for the Bode sensitivity. The merit of the relative sensitivity definition is that it is always dimensionless. A useful feature of the Bode sensitivity is that it applies both to system parameter sensitivity and system sensitivity with respect to a subsystem, say G_0, by the definition below:

Definition (Bode's Sensitivity Function): *Let* $G = G(s, \boldsymbol{\alpha})$ *and* $G_0 = G(s, \boldsymbol{\alpha}_0)$ *be the actual (or nominal) transfer functions respectively with* $\boldsymbol{\alpha}_0$ *as the nominal parameter vector. The partial derivative:*

$$S_{\alpha_j}^{G} = \left.\frac{\partial G / G}{\partial \alpha_j / \alpha_j}\right|_{\boldsymbol{\alpha}_0} = \left.\frac{\partial G}{\partial \alpha_j}\right|_{\boldsymbol{\alpha}_0} \cdot \frac{\alpha_{j0}}{G_0} \quad (4.11)$$

is called the **sensitivity function of Bode** *or the* **classical sensitivity function**. (Frank, 1978) (p. 49).

Note that this is equivalent to the relative sensitivity function defined earlier.

Often the Bode sensitivity is written as:

$$\overline{\mathbf{S}}_p^f = \left.\frac{\partial \ln(f)}{\partial \ln(p)}\right|_{p_0} \quad (4.12)$$

This notation is inappropriate in the context of this thesis as the logarithm is dimensionally undefined if it does not act in-ratio with another parameter (See the discussion in Chapter 2 on the dimensional requirements of transcendental functions). This requirement is somewhat blurred in the mathematical treatment, and therefore this log-style notation will be strongly discouraged.

A useful tool in the calculation of Bode sensitivity functions is the general form of the sensitivity for a transfer function.

$$G(s, \alpha) = \frac{N(s, \alpha)}{D(s, \alpha)} \quad (4.13)$$

Dropping the s-notation, we can write:

$$
\begin{aligned}
S_\alpha^G &= \frac{\alpha}{G}\frac{\partial}{\partial\alpha}\left(\frac{N}{D}\right) \\
&= \alpha\cdot\frac{D}{N}\cdot\left[\frac{D\cdot\dfrac{\partial N}{\partial\alpha}-N\cdot\dfrac{\partial D}{\partial\alpha}}{D^2}\right] \\
&= \frac{\alpha}{N}\cdot\frac{\partial N}{\partial\alpha}-\frac{\alpha}{D}\cdot\frac{\partial D}{\partial\alpha} \\
S_\alpha^G &= S_\alpha^N - S_\alpha^D
\end{aligned}
\tag{4.14}
$$

Given a transfer function separated into terms dependent on a parameter (or subsystem) and systems not dependent on the parameter (or subsystem):

$$
G(s,\alpha)=\frac{G_1(s)+\alpha\cdot G_2(s)}{G_3(s)+\alpha\cdot G_4(s)} \tag{4.15}
$$

Using the result for fractional forms developed earlier, the sensitivity of this representation with respect to the parameter (or subsystem) is given as:

$$
S_\alpha^G = \left.\frac{\alpha\left(G_1G_2-G_1G_4\right)}{\left(G_3+\alpha\cdot G_4\right)\cdot\left(G_1+\alpha\cdot G_2\right)}\right|_{\alpha_0} \tag{4.16}
$$

This serves to simplify many sensitivity calculations in later sections.

One drawback of the sensitivity definition of Bode is that it is strictly defined only for infinitesimal parameter perturbations. The sensitivity definition of Horowitz and the comparison sensitivity definition of Perkins and Cruz both address this problem. Both methods are defined for moderate system perturbations, and it can be shown (Frank, 1978) that they yield almost equivalent sensitivity definitions.

4.3 Sensitivity Invariance by Dimensional Analysis

The introduction to the chapter presented the argument that, for the spatial-parametric problems of fluid dynamics and heat transfer, dimensional analysis is commonly used to collapse the parameter dependence of system descriptions. In the common usage within these fields, a statement of dynamic similitude implies that one system is experimentally equivalent to another. The exact mathematical meaning of this statement and its relevance to system properties from an control-theoretic viewpoint is somewhat vague. Therefore, the basic mathematical principles of dimensional analysis are examined, specifically Euler's Homogenous Function Theorem which is commonly used to prove the Pi Theorem. This statement is shown to provide the number and form of sensitivity invariants of the governing physical equation, where it is assumed that the governing equation is homogenous (i.e. a physical equation as per the definition of Chapter 2). Elimination of the sensitivity interdependence is shown to produce precisely the result of the Pi-Theorem, and it is revealed that the statement of dynamic similitude is actually a strong statement regarding equivalency of sensitivities between systems.

4.3.1 Euler's Homogenous Function Theorem: Sensitivity Implications

The relationship between dimensional analysis and sensitivity theory was first probed by Euler in the late 1700's, and a result of his analysis was the important Euler's Homogenous Function Theorem (EHF Theorem) which is also known as Euler's Theorem of Homogenous Functions. The derivation of the EHF Theorem follows below, and this form was developed from a methodology originally presented by Langhaar (Langhaar, 1951)(p. 153-4). It has modified for ease of presentation.

To begin the discussion, we must first describe the meaning of a 'homogenous function'. A homogenous function is one whose output becomes scaled by a factor, k^n, when each of the arguments to the function are scaled by the factor k. The exponent n is called the order of the homogeneity; for instance, a homogenous system description with $n = 2$ called homogenous of order 2. To illustrate different levels of homogeneity, consider a function of two variables: $y = f(x_1, x_2)$. If $y = f(x_1, x_2)$ is of the form:

$$y = 7 \cdot x_1 + 2 \cdot x_2 \tag{4.17}$$

Then the system is homogenous of order 1, because

$$\begin{aligned} f(k \cdot x_1, k \cdot x_2) &= 7 \cdot k \cdot x_1 + 2 \cdot k \cdot x_2 \\ &= k \cdot (7 \cdot x_1 + 2 \cdot x_2) \\ &= k \cdot y \end{aligned} \tag{4.18}$$

If $y = f(x_1, x_2)$ is of the form:

$$y = 7 \cdot x_1 / x_2 \tag{4.19}$$

Then the system is homogenous of order 0, because

$$\begin{aligned} f(k \cdot x_1, k \cdot x_2) &= 7 \cdot k \cdot x_1 / k \cdot x_2 \\ &= 7 \cdot x_1 / x_2 \\ &= y \end{aligned} \tag{4.20}$$

Most functions are not homogenous, and to illustrate this consider $y = f(x_1, x_2)$ of the form:

$$y = 7 \cdot x_1 + x_2^3 \tag{4.21}$$

It is not homogenous because no power of k can be factored from the equation.

With the notion of homogeneity well-defined, we present Euler's homogenous function theorem which is central to the discussion of this chapter. It is a statement regarding functions that exhibit different homogeneity to different variable scaling factors. Given a function of the form:

$$y = f(x_1, x_2, \ldots, x_n) \tag{4.22}$$

that is homogenous with respect to constants, A, B, C, we can write:

$$K \cdot y = f(K_1 \cdot x_1, K_2 \cdot x_2, \ldots, K_n \cdot x_n) \tag{4.23}$$

where the constants K and K_i are constrained by:

$$\begin{aligned} K &= A^a B^b C^c \\ K_i &= A^{a_i} B^{b_i} C^{c_i} \end{aligned} \tag{4.24}$$

and the values of $a_i, b_i, c_i, ..., a, b, c...$ are fixed by the governing equation. Differentiation of Equation 4.23 with respect to A gives:

$$\underbrace{\frac{\partial y}{\partial A}}_{1} = \underbrace{\underbrace{\frac{\partial y}{\partial x_1}}_{2} \cdot \underbrace{\frac{\partial x_1}{\partial A}}_{3}}_{4} + \frac{\partial y}{\partial x_2} \cdot \frac{\partial x_2}{\partial A} + \ldots + \frac{\partial y}{\partial x_n} \cdot \frac{\partial x_n}{\partial A} \tag{4.25}$$

Examining this term-by-term, we can rewrite this equation while substituting in the dependence on K as specified in Equation 4.23:

$$\text{Term 1} \qquad \begin{aligned} \frac{\partial(K\cdot y)}{\partial A} &= y\frac{\partial K}{\partial A} \\ &= y\frac{\partial\left(A^{a}B^{b}C^{c}\right)}{\partial A} \\ &= a\cdot y\cdot A^{a-1}\cdot B^{b}C^{c} \\ &= a\cdot y\cdot\frac{K}{A} \end{aligned} \tag{4.26}$$

$$\text{Term 2} \quad \frac{\partial(K\cdot y)}{\partial\left(K_1\cdot x_1\right)} = \frac{K}{K_1}\frac{\partial y}{\partial x_1} \tag{4.27}$$

$$\text{Term 3} \qquad \begin{aligned} \frac{\partial(K_1\cdot x_1)}{\partial A} &= x_1\cdot\frac{\partial\left(A^{a_1}B^{b_1}C^{c_1}\right)}{\partial A} \\ &= a_1\cdot x_1\cdot A^{a_1-1}\cdot B^{b_1}C^{c_1} \\ &= a_1\cdot x_1\cdot\frac{K_1}{A} \end{aligned} \tag{4.28}$$

$$\text{Term 4} \quad \begin{aligned} \frac{\partial\left(Ky\right)}{\partial\left(K_1x_1\right)}\cdot\frac{\partial\left(K_1x_1\right)}{\partial A} &= \frac{K}{K_1}\cdot\frac{K_1}{A}a_1\cdot x_1\cdot\frac{\partial y}{\partial x_1} \\ &= \frac{K}{A}a_1\cdot x_1\cdot\frac{\partial y}{\partial x_1} \end{aligned} \tag{4.29}$$

The remaining terms can be determined from the above operations. Equation 4.25 then becomes:

$$a\cdot y\cdot\frac{K}{A} = \frac{K}{A}a_1\cdot x_1\cdot\frac{\partial y}{\partial x_1} + \frac{K}{A}a_2\cdot x_2\cdot\frac{\partial y}{\partial x_2} + \ldots + \frac{K}{A}a_n\cdot x_n\cdot\frac{\partial y}{\partial x_n} \tag{4.30}$$

or:

$$a\cdot y = a_1\cdot x_1\cdot\frac{\partial y}{\partial x_1} + a_2\cdot x_2\cdot\frac{\partial y}{\partial x_2} + \ldots + a_n\cdot x_n\cdot\frac{\partial y}{\partial x_n} \tag{4.31}$$

Similarly, we can differentiate with respect to B and C to complete a direct proof of the following theorem:

Theorem: Euler's Homogenous Function Theorem (the EHF Theorem): *A function satisfying Equation* 4.23 *and* 4.24*, whose parameter dependence is differentiable, is also a solution to the following equation:*

$$\begin{aligned} a \cdot y &= a_1 \cdot x_1 \cdot \frac{\partial y}{\partial x_1} + a_2 \cdot x_2 \cdot \frac{\partial y}{\partial x_2} + \ldots + a_n \cdot x_n \cdot \frac{\partial y}{\partial x_n} \\ b \cdot y &= b_1 \cdot x_1 \cdot \frac{\partial y}{\partial x_1} + b_2 \cdot x_2 \cdot \frac{\partial y}{\partial x_2} + \ldots + b_n \cdot x_n \cdot \frac{\partial y}{\partial x_n} \\ c \cdot y &= c_1 \cdot x_1 \cdot \frac{\partial y}{\partial x_1} + c_2 \cdot x_2 \cdot \frac{\partial y}{\partial x_2} + \ldots + c_n \cdot x_n \cdot \frac{\partial y}{\partial x_n} \end{aligned} \qquad (4.32)$$

Note that the converse of this theorem, that equations of form 4.32 imply an equivalence of the form 4.23 and 4.24, has also been historically proven but is not presented here.

The importance of the EHF Theorem is that it applies to every physical model because true physical descriptions must also be dimensionally homogenous. For instance, given a general form of a physical equation:

$$f\left(x_1, x_2, \ldots, x_n\right) = y \qquad (4.33)$$

The model output and model parameters will be measured in terms of physical dimensions (units). Each of the physical dimensions of $y, x_1 \ldots x_n$ is given by rational powers of fundamental dimensions, $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3, \ldots$. For instance, let us suppose that the physical dimensions of y are known and given by, $y \doteqdot \mathrm{d}_1^a \cdot \mathrm{d}_2^b \cdot \mathrm{d}_3^c, \ldots$, and that the arguments of the function (parameters) share similar unit dependence except with different exponents. We denote the argument exponents using subscripts for each argument. The dimensional exponents of each parameter are then summarized by the columns of the following dimensional matrix:

$$\begin{array}{c|c:cccc} & y & x_1 & x_2 & \ldots & x_n \\ \hline \mathrm{d}_1 & a & a_1 & a_2 & & a_n \\ \mathrm{d}_2 & b & b_1 & b_2 & & b_n \\ \mathrm{d}_3 & c & c_1 & c_2 & & c_n \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \end{array} \qquad (4.34)$$

The system model of Equation 4.33, dependent on the above parameters, satisfies the conditions of the EHF Theorem. To show this, we allow a simple change in dimensional basis as described in Chapter 3:

$$\begin{aligned} \mathrm{d}_1 &= k_1 \cdot \mathrm{D}_1 \\ \mathrm{d}_2 &= k_2 \cdot \mathrm{D}_2 \\ &\vdots \\ \mathrm{d}_d &= k_d \cdot \mathrm{D}_d \end{aligned} \qquad (4.35)$$

For instance, the problem might require three fundamental dimensions, $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3$, with units of kilometers-hours-kilograms. The new unit system, $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3$, may be meters-seconds-grams and so the values, k_1, k_2, k_3, become respectively $1000, 3600, 1000$. Under such changes in the unit system, Equation 4.33 becomes:

$$K \cdot y = f\left(K_1 \cdot x_1, K_2 \cdot x_2, \ldots, K_n \cdot x_n\right) \qquad (4.36)$$

with the parameters given by:

$$\begin{aligned} K &= k_1^{\,a} k_2^{\,b} k_3^{\,c} \ldots \\ K_i &= k_1^{\,a_i} k_2^{\,b_i} k_3^{\,c_i} \ldots, \quad i = 1 : N_p \end{aligned} \qquad (4.37)$$

where N_p is the number of parameters. Equations 4.38 and 4.39 show that the original equation will always exhibit homogeneity with respect to the dimension system used (it inferred that this is why physical equations are called 'dimensionally homogenous'). The system representation given by Equation 4.33 is sufficiently general to represent all physical system equations, and therefore the statement of the theorem applies to all mathematical system representations used in control theory to model physical systems under the assumptions of the existence of the partial derivatives.

The EHF Theorem also provides a compact proof to the Pi Theorem. If we apply the results of the EHF Theorem to the general equation form for a generalized physical model in Equation 4.33, the following set of linear equations is obtained:

$$y\begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \cdot \frac{\partial y}{\partial x_1} \\ x_2 \cdot \frac{\partial y}{\partial x_2} \\ \vdots \\ x_n \cdot \frac{\partial y}{\partial x_n} \end{bmatrix} \tag{4.40}$$

Note that the middle matrix is a form of the dimensional matrix described in Chapter 3. Given the assumptions on rank as discussed in Chapter 3, then for any fixed mapping from input variables to output variables, the minimal number of solutions to this equation occur if, and only if, the terms $a, b, c \ldots = 0$. Under this assumption, linear algebra predicts that the number of solutions is given by $n-d$, where n is the number of parameters and d is the number of dimensions spanned by the parameters. Thus, there are $n-d$ solutions that are dimensionally invariant for a system of n parameters and d independent dimensions. This completes a proof of Buckingham's Theorem originally presented by Buckingham and others.

4.3.2 Sensitivity Invariants Due to Dimensional Analysis

We may immediately note that Equation 4.40 makes a direct statement on the relative sensitivities of a physical equation. Specifically, if we divide through by the term y we obtain:

$$\begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} \frac{x_1}{y} \cdot \frac{\partial y}{\partial x_1} \\ \frac{x_2}{y} \cdot \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{x_n}{y} \cdot \frac{\partial y}{\partial x_n} \end{bmatrix} \tag{4.41}$$

This can be rewritten in terms of Bode sensitivities as:

$$\begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} S_{x_1}^{y} \\ S_{x_2}^{y} \\ \vdots \\ S_{x_n}^{y} \end{bmatrix} \qquad (4.42)$$

The right-hand column vector represents Bode sensitivities of the system with respect to each of the model parameters, and the choice of the measure y can be any measure of the system. For instance, y may be a model output, y(t), a model transfer function G(s), or an internal model state, x(t) or x(s), or a control input u(s) or u(t). Therefore, the following general statement can be made on physical models, summarizing the primary point of this chapter:

The Bode sensitivities of the parameters of a physical equation are <u>not</u> *independent if the governing equation satisfies the assumptions of Euler's Homogenous Function Theorem. The interdependence represents a sensitivity invariance of the system, and the form of the invariance is* **always** *a set of linear equations, even if the governing model is nonlinear.*

In the literature (Frank, 1978; Eslami, 1994), it is has been known since the mid 1960's that sensitivity measures for circuit networks exhibit invariant properties. However, these properties were generally associated only with a specific sensitivity measure, for instance the Total Sensitivity Function (see Eslami, Section 3.6), or associated only with specific subclasses of physical systems, for instance the sensitivity of voltages inside circuits consisting solely of diodes, resistors, capacitors, and inductors (the presentation of Frank).

It is clear from the proofs of these sensitivity invariants that it is not recognized in the control community that sensitivity invariance is dependent on dimensional homogeneity and provable by the EHF Theorem. From the literature surveyed, it appears that it has not heretofore been recognized that all physical systems may exhibit sensitivity invariants with respect to any measure. For instance, the sensitivity invariants of the circuit analysis of the 1960's only capture one of the potentially three subspaces of sensitivity invariants of generalized RLC circuits, and even in this case the proofs are in general valid only for linear elements within the circuit. Some of these proofs (e.g. Frank's presentation, p. 155-6) are interestingly based on modified forms of the EHF Theorem previously presented.

The observation that the invariant subspace is represented by a set of linearly dependent sensitivity functions extends nicely within the concepts of sensitivity analysis. It is well known that sensitivity analysis of linear and nonlinear system equations will always generate linear sensitivity functions under very inclusive assumptions on the system (Frank, 1978). With an additional linear constraint between the sensitivities, we find that there cannot be a nonlinear coupling between the sensitivity functions. Thus, the geometric interpretation of a Jacobian operation and other implicit sensitivity calculations as a linear operator is generally preserved.

4.4 Four Examples of Sensitivity Invariance

The understanding of the mathematical statement of the previous subsection is essential to the remainder of the thesis, and so several examples were chosen to illustrate its validity. These examples are chosen to exhibit a range of analysis to demonstrate the scope of the previous claims:

(1) a static mapping
(2) a open-loop time-domain response
(3) an open-loop frequency-domain response
(4) a closed-loop frequency domain response

Additional examples extending this result (but not in a sensitivity context) to partial differential equations and/or nonlinear systems can be found in almost any dimensional analysis literature.

4.4.1 Static Mapping Sensitivity: The Period of a Pendulum

The period of a point-mass pendulum is given by the following static mapping:

$$T = 2 \cdot \pi \cdot \sqrt{\frac{h}{g}} \tag{4.43}$$

This equation can be rewritten as:

$$T = f\left(h, g\right) \tag{4.44}$$

Each of the variables of the equation are listed below with their physical dimensions:

Variable	Symbol	Dimension
Period of oscillation	T	s
Length of suspension	h	m
Gravitational constant	g	$\mathrm{m \cdot s^{-2}}$

If we form a dimensional matrix of the above information, we find that:

$$\begin{array}{r|c|cc} & T & h & g \\ \hline \text{meters} & 0 & 1 & 1 \\ \text{seconds} & 1 & 0 & -2 \end{array} \tag{4.45}$$

Comparing this to the form required by the EHF Theorem, we find that:

$$\begin{array}{lll} a = 0 & a_1 = 1 & a_2 = 1 \\ b = 1 & b_1 = 0 & b_2 = -2 \end{array} \tag{4.46}$$

Application of the EHF Theorem yields the following relationships between the sensitivities:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} S_h^T \\ S_g^T \end{bmatrix} \tag{4.47}$$

Note that this equation is solvable for the sensitivities, and we obtain from Equation 4.42:

$$\begin{bmatrix} S_h^T \\ S_g^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \tag{4.48}$$

We can algebraically confirm this relationship by calculating the Bode sensitivities directly from the governing equation:

$$\begin{aligned} \frac{\partial T}{\partial h} &= \pi \cdot \frac{h^{-\frac{1}{2}}}{g^{1/2}} \\ S_h^T &= \frac{h}{T} \cdot \frac{\partial T}{\partial h} \\ &= h \cdot \underbrace{\frac{g^{1/2}}{2 \cdot \pi \cdot h^{1/2}}}_{1/T} \cdot \pi \cdot \frac{h^{-\frac{1}{2}}}{g^{1/2}} \\ &= 1/2 \end{aligned} \tag{4.49}$$

and:

$$\frac{\partial T}{\partial g} = -\pi \cdot \frac{h^{\frac{1}{2}}}{g^{3/2}}$$

$$S_g^T = \frac{g}{T} \cdot \frac{\partial T}{\partial g}$$

$$= g \cdot \underbrace{\frac{g^{1/2}}{2 \cdot \pi \cdot h^{1/2}}}_{1/T} \cdot -\pi \cdot \frac{h^{\frac{1}{2}}}{g^{3/2}}$$

$$= -1/2$$

This numerically confirms the original mathematical statement.

The previous example illustrates that the knowledge of the sensitivity invariants determined by dimensional analysis were sufficient to completely determine the sensitivities to all parameters. No additional calculations or system measurements were needed. If one calculates the pi-parameters, one finds that there is only one pi-parameter for this system, namely

$$\pi_1 = T \cdot \sqrt{\frac{g}{h}} \qquad (4.50)$$

Analysis of this problem should reveal that equations of only one pi-parameter share this important property, that the equations of the invariant sensitivity subspace are sufficient to completely determine Bode sensitivities of all parameters of the equation without additional knowledge of the system.

4.4.2 Open-Loop Time Domain Response Sensitivity: Nonlinear System

Given the following nonlinear differential equation:

$$\frac{dx}{dt} = -b \cdot x + a \cdot x^2 \qquad (4.51)$$

With the nominal, numerical value of b and a equal to 1. The solution to the differential equation is:

$$x(t)=\frac{\frac{b}{a}\cdot x_0\cdot e^{-bt}}{\frac{b}{a}-x_0+x_0\cdot e^{-bt}} \tag{4.52}$$

If we presuppose that the state x(t) has fundamental dimensions of some unit, say meters, then the equation imposes the following dimension constraints:

Variable	**Symbol**	**Dimension**
State	x	m
Time	t	s
Initial position	x_0	m
Time constant	b	s^{-1}
Scaling factor	a	$m^{-1}\cdot s^{-1}$

If we form a dimensional matrix of for the above parameters, we find that:

$$\begin{array}{r|c|cccc} & x & t & x_0 & b & a \\ \hline \text{meters} & 1 & 0 & 1 & 0 & -1 \\ \text{seconds} & 0 & 1 & 0 & -1 & -1 \end{array} \tag{4.53}$$

Comparing this to the form required by the EHF Theorem, we find that:

$$\begin{array}{lllll} a=1 & a_1=0 & a_2=1 & a_3=\ 0 & a_4=-1 \\ b=0 & b_1=1 & b_2=0 & b_3=-1 & b_4=-1 \end{array} \tag{4.54}$$

Application of the EHF Theorem then yields the following relationships between the sensitivities:

$$\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & -1\\ 1 & 0 & -1 & -1\end{bmatrix}\cdot\begin{bmatrix}S_t^x\\ S_{x_0}^x\\ S_b^x\\ S_a^x\end{bmatrix} \tag{4.55}$$

Note that this equation is not solvable for the sensitivities using the invariant spaces alone as in the previous example. However, we can again algebraically confirm this relationship by calculating the Bode sensitivities directly from the governing equation. If we represent the numerator and denominator of the original system of Equation 4.52 as:

$$N = \frac{b}{a} \cdot x_0 \cdot e^{-bt}, D = \frac{b}{a} - x_0 + x_0 \cdot e^{-bt} \qquad (4.56)$$

The following sensitivity functions are obtained:

$$S_t^x = \frac{-b \cdot t \cdot D + b \cdot t \cdot x_0 \cdot e^{-b \cdot t}}{D} \qquad (4.57)$$

$$S_{x_0}^x = \frac{D - x_0 \cdot e^{-b \cdot t} + x_0}{D} \qquad (4.58)$$

$$S_b^x = \frac{(1 - b \cdot t) \cdot D - \frac{b}{a} + b \cdot t \cdot x_0 \cdot e^{-b \cdot t}}{D} \qquad (4.59)$$

$$S_a^x = \frac{-D + \frac{b}{a}}{D} \qquad (4.60)$$

If we substitute these sensitivities into Equation 4.42 to confirm the validity, we obtain:

$$\begin{aligned} S_{x_0}^x - S_a^x &= \frac{D - x_0 \cdot e^{-b \cdot t} + x_0}{D} + \frac{+D - \frac{b}{a}}{D} \\ &= \frac{D - D + D}{D} = 1 \end{aligned} \qquad (4.61)$$

$$\begin{aligned} S_t^x - S_b^x - S_a^x &= \frac{-b \cdot t \cdot D + b \cdot t \cdot x_0 \cdot e^{-b \cdot t}}{D} - \frac{(1 - b \cdot t) \cdot D - \frac{b}{a} + b \cdot t \cdot x_0 \cdot e^{-b \cdot t}}{D} - \frac{-D + \frac{b}{a}}{D} \\ &= \frac{\left(-b \cdot t \cdot D + b \cdot t \cdot D\right) + \left(b \cdot t \cdot x_0 \cdot e^{-b \cdot t} - b \cdot t \cdot x_0 \cdot e^{-b \cdot t}\right) + (D - D) + \left(-\frac{b}{a} + \frac{b}{a}\right)}{D} \\ &= 0 \end{aligned}$$

This numerically confirms the original mathematical statement. This example illustrates that nonlinear equations also produce linear sensitivity invariants and that system equations must satisfy sensitivity invariant relationships. Specifically, we may write:

$$\begin{aligned} S_t^x &= \frac{t}{x(t)}\frac{dx}{dt} = S_b^x + S_a^x \\ \Rightarrow \frac{dx}{dt} &= \frac{x}{t}\cdot\left(S_b^x + S_a^x\right) \end{aligned} \tag{4.62}$$

Since time derivatives that do not use a dimensionless time will always require time units, equations of the form 4.42 will generally exist for physical system representations.

4.4.3 Open-Loop Frequency-Domain Sensitivity: Mass-Spring Damper

A well-known a physical example used throughout control theory is the system of the mass-spring-damper. The governing system equation is given by:

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt^2} + Kx = U(t) \tag{4.63}$$

where the control input, $U(t)$, is a time varying force. The above equation is represented in the frequency domain via the Laplace transform (assuming zero initial conditions):

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{m\cdot s^2 + \beta\cdot s + k} \tag{4.64}$$

It is now demonstrated that the sensitivity functions of this system exhibit invariance as given by the EHF Theorem. We will suppose that the dimensions of each of the system parameters have the standard definition:

Variable	Symbol	Dimension
System	$G(s)$	$\text{kg}^{-1}\cdot s^2$
Mass	m	kg
Damping	β	$\text{kg}\cdot s^{-1}$
Spring constant	k	$\text{kg}\cdot s^{-2}$
Laplace variable	s	s^{-1}

Note that the Laplace variable is distinguished from time by the use of italics for the variable and standard typeface for the dimension. If we form a dimensional matrix of for the above parameters, we find that:

$$\begin{array}{r|r|rrrr} & G(s) & m & \beta & k & s \\ \hline \text{kilograms} & -1 & 1 & 1 & 1 & 0 \\ \text{seconds} & 2 & 0 & -1 & -2 & -1 \end{array} \tag{4.65}$$

Comparing this to the form of the EHF Theorem, we find that:

$$\begin{array}{lllll} a=-1 & a_1=1 & a_2=\ \ 1 & a_3=\ \ 1 & a_4=\ \ 0 \\ b=\ \ 2 & b_1=0 & b_2=-1 & b_3=-2 & b_4=-1 \end{array} \tag{4.66}$$

Application of the EHF Theorem yields the following relationships between the sensitivities:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} S_m^G \\ S_\beta^G \\ S_k^G \\ S_s^G \end{bmatrix} \tag{4.67}$$

Again, we can algebraically confirm this relationship by calculating the Bode sensitivities directly from the governing equation. The following sensitivity functions are obtained:

$$S_m^G = -s^2 \cdot m_0 \cdot G_0 \tag{4.68}$$

$$S_\beta^G = -s \cdot \beta \cdot G_0 \tag{4.69}$$

$$S_k^G = -k_0 \cdot G \tag{4.70}$$

$$S_s^G = \left(-2 \cdot m \cdot s^2 - \beta \cdot s\right) \cdot G \tag{4.71}$$

If we substitute these sensitivities into Equation 4.42 to confirm the validity, we obtain:

$$\begin{aligned} S_m^G + S_\beta^G + S_k^G &= -s^2 \cdot m_0 \cdot G_0 - s \cdot \beta \cdot G_0 - k_0 \cdot G \\ &= -(s^2 \cdot m_0 + s \cdot \beta + k_0) \cdot G \\ &= -G^{-1} \cdot G = -1 \end{aligned} \tag{4.72}$$

$$-S_{\beta}^{G} - 2 \cdot S_{k}^{G} - S_{s}^{G} = -1 \cdot \left(-s \cdot \beta \cdot G_0\right) - 2 \cdot \left(-k_0 \cdot G\right) - 1 \cdot \left(-2 \cdot m \cdot s^2 \cdot G - \beta \cdot s \cdot G\right)$$
$$= G_0 \cdot \left(2 \cdot m \cdot s^2 + 2s \cdot \beta + 2k_0\right)$$
$$= 2 \cdot G_0 \cdot G_0^{-1} = 2$$

This numerically confirms the original mathematical statement. This example illustrates that the sensitivity invariants apply to the frequency domain and that sensitivity to the frequency variable, s (or jw in the case of a frequency response) must be considered for calculation of the sensitivity invariants.

4.4.4 Closed-Loop Frequency-Domain Sensitivity: DC Motor Control

As a final example, consider the sensitivity invariant equations of a closed-loop system. For this example, we consider a model of a DC motor with inductance ignored. The velocity is controlled via a proportional controller in the feedback loop. The math model is given by the following physical equation.

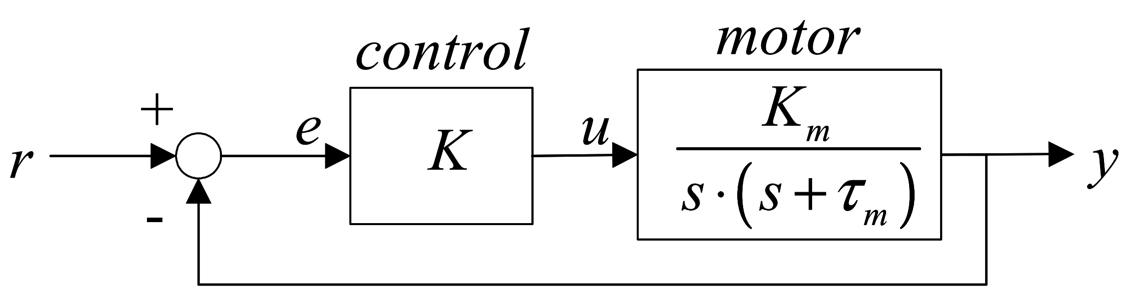


Figure 4.6: A simple position control loop for the DC motor

The amplifier (whose model is absorbed into the motor transfer function) transforms a control input with dimensions of velocity into an appropriate voltage, and this implicit dimensional transformation is absorbed into the motor gain K_m. The closed-loop system equation is then given by:

$$G(s) = \frac{K \cdot K_m}{s^2 + \tau_m \cdot s + K \cdot K_m} \qquad (4.73)$$

where the control-loop input is a reference position output is the actual position. The physical dimensions of each of the equation parameters are given by the following table:

Variable	Symbol	Dimension
System	$G(s)$	1
Proportional gain	K	$\mathrm{volts}\cdot\mathrm{m}^{-1}$
Motor gain	K_m	$\mathrm{volts}^{-1}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$
Motor time constant	τ_m	s^{-1}
Laplace variable	s	s^{-1}

Again, note that the Laplace variable is distinguished from time by the use of italics for the variable and standard typeface for the dimension. If we form a dimensional matrix of for the above parameters, we find that:

$$\begin{array}{r|c|ccccc} & G & K & K_m & \tau_m & s \\ \hline \text{meters} & 0 & -1 & 1 & 0 & 0 \\ \text{seconds} & 0 & 0 & -2 & -1 & -1 \\ \text{volts} & 0 & 1 & -1 & 0 & 0 \end{array} \qquad (4.74)$$

Comparing this to the form of the EHF Theorem, we find that:

$$\begin{array}{lllll} a=0 & a_1=-1 & a_2=\ \ 1 & a_3=\ \ 0 & a_4=\ \ 0 \\ b=0 & b_1=\ \ 0 & b_2=-2 & b_3=-1 & b_4=-1 \\ c=0 & c_1=\ \ 1 & c_2=-1 & c_3=\ \ 0 & c_4=\ \ 0 \end{array} \qquad (4.75)$$

Application of the EHF Theorem yields the following relationships between the sensitivities:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} S_K^G \\ S_{K_m}^G \\ S_{\tau_m}^G \\ S_s^G \end{bmatrix} \qquad (4.76)$$

Note that the first equation for sensitivity invariance is equivalent to the last equation for sensitivity invariance. Again, we can algebraically confirm this relationship by calculating the

Bode sensitivities directly from the governing equation. The following sensitivity functions are obtained:

$$S_K^G = \left(1 - G_0\right) \tag{4.77}$$

$$S_{K_m}^G = \left(1 - G_0\right) \tag{4.78}$$

$$S_{\tau_m}^G = -G_0 \cdot \frac{\tau_m \cdot s}{K^2 \cdot K_m^2} \tag{4.79}$$

$$S_s^G = G_0 \cdot \frac{-2 \cdot s^2 - \tau_m \cdot s}{K^2 \cdot K_m^2} \tag{4.80}$$

If we substitute these sensitivities into Equation 4.76 to confirm the validity, we obtain:

$$-S_K^G + S_{K_m}^G = 0 \tag{4.81}$$

$$\begin{aligned}
-2 \cdot S_{K_m}^G - S_{\tau_m}^G - S_s^G &= -2 \cdot \left(1 - G_0\right) - 1 \cdot \left(-G_0 \cdot \frac{\tau_m \cdot s}{K^2 \cdot K_m^2}\right) - 1 \cdot \left(G_0 \cdot \frac{-2 \cdot s^2 - \tau_m \cdot s}{K^2 \cdot K_m^2}\right) \\
&= -2 + G_0 \cdot \left(2 + \frac{2 \cdot s^2 + 2 \cdot \tau_m \cdot s}{K^2 \cdot K_m^2}\right) \\
&= -2 + G_0 \cdot \left(\frac{2}{G_0}\right) = 0
\end{aligned}$$

Once again, this numerically confirms the original mathematical statement. This example illustrates that sensitivity analysis of the closed-loop is feasible, and it is also meant to demonstrate a sensitivity aspect used in the remainder of this chapter. Specifically, the control gain is dimensionless, and for this reason the Bode sensitivity to gain does not appear in the calculations for the invariant subspaces. This property will be used in the next section to simplify sensitivity dependence of system dynamics by a reparameterization of the governing model.

4.5 Sensitivity Decoupling by Dimensionless Reparameterization

The previous section demonstrated that equations for physical representations of systems generally may have one or several sensitivity invariants. The last example of the previous section demonstrated, however, that a dimensionless term will be excluded from the equations for the sensitivity invariant. This notion is developed further in this section by a method of model reparameterization that can completely eliminate sensitivity invariants generated by dimensional analysis (although others may be present and are discussed later). The result of this reparameterization is precisely the dimensionless system model representation based on pi-parameters developed in Chapter 3.

To review, given an arbitrary system representation as in the previous section:

$$y = f\left(x_1, x_2, \ldots, x_n\right) \tag{4.82}$$

The application of the EHF Theorem generates invariant sensitivity subspaces according to equations of the form:

$$\begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\mathbf{D}} \cdot \begin{bmatrix} S_{x_1}^{y} \\ S_{x_2}^{y} \\ \vdots \\ S_{x_n}^{y} \end{bmatrix} \tag{4.83}$$

The matrix elements for the left-hand column vector and the center matrix are given by dimensional exponents for the parameters of the system equation. Each row represents a different sensitivity subspace that is invariant by dimensional analysis. If we take an arbitrary row of the above matrix, we obtain an equation of the form:

$$j = \begin{bmatrix} j_1 & j_2 & \cdots & j_n \end{bmatrix} \cdot \begin{bmatrix} S_{x_1}^{y} \\ S_{x_2}^{y} \\ \vdots \\ S_{x_n}^{y} \end{bmatrix} \tag{4.84}$$

Note that the values of the terms $j_1 \quad j_2 \quad \ldots \quad j_n$ depend on the parameters $x_1, x_2, \ldots, x_n$.

We now consider methods to reduce or eliminate the sensitivity invariant equations. Let us specifically consider changing our measurement system in ways that eliminate or simplify the sensitivity equations of the form of Equation 4.84. One might think of this task as attempting to choose the 'best' unit system for the problem under consideration. For instance, if our equation has several length measurements of size L, we may choose to measure these lengths by feet or meters or light-years. However, if we are completely free to choose the measurement system, we should choose a special 'length' stick of length L. In this way, we obtain unity for each measurement of L! This greatly simplifies the governing equation because the parameter of L seems to 'drop' out and would no longer appear in our governing relationship or model. However, we must remember to be consistent, that all other parameters also must be measured in length units of L. In essence, we must multiply and divide each of the remaining parameters by L so that their distance measurements remain consistent.

Mathematically, the method is given as follows. Given a row, $j_1 \quad j_2 \quad \dots \quad j_n$, of Equation 4.83, choose one of the parameters corresponding to a non-zero value. Let us call this parameter the kth parameter. The remaining parameters must be measured with respect to this kth parameter. Thus, from the row of the sensitivity equation, determine which of the terms, $j_1 \quad j_2 \quad \dots \quad j_n$ that are also nonzero other than the parameter associated with k. Reparameterize the each of the associated parameters by a transformation based on the kth variable to a power:

$$p_i' = p_i \cdot \left(p_j\right)^{-j_i/j_k} \tag{4.85}$$

The result will eliminate both the kth parameter from the system model as well as eliminate one of the subspaces of sensitivity invariance. The following example taken from the previous section illustrates this technique:

Consider again the mass-spring-damper presented previously. The governing system equation is given as:

$$G(s) = \frac{1}{m \cdot s^2 + \beta \cdot s + k} \tag{4.86}$$

and application of the EHF Theorem yields the following relationships between the sensitivities:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} S_m^G \\ S_\beta^G \\ S_k^G \\ S_s^G \end{bmatrix} \tag{4.87}$$

Now let us say we wish to eliminate the first row representing a subspace of sensitivity invariance. The first row is a statement of the coupling of the sensitivities of the parameters, G, m, β, k, due to the units of mass measurement. . While we can choose any one of the parameters G, m, β, k to eliminate from the sensitivity equation, we cannot not eliminate s in this case because it has a zero coefficient in the first row and therefore does not enter the invariance relationship given by the choice of mass units.

Let us (arbitrarily) choose to eliminate the mass, m, from the equation. By the method previously described, we choose a reparameterization (primed variables) of the remaining terms in this row G, β, k such that the variable m is eliminated. Equation 4.85 gives the required transformations:

$$\begin{aligned} G' &= G(s) \cdot m^{-(-1/1)} = G(s) \cdot m \\ m' &= m \cdot m^{-(1/1)} = 1 \\ \beta' &= \beta \cdot m^{-(1/1)} = \beta / m \\ k' &= k \cdot m^{-(1/1)} = k / m \end{aligned} \tag{4.88}$$

Note that the mass term is now implicitly used to measure each of the reparameterized variables (represented by a prime). The reparameterized (prime) system is given by:

$$G'(s) = \frac{1}{s^2 + \beta' \cdot s + k'} \tag{4.89}$$

and mass no longer *explicitly* enters the system equations. If we examine the dimensions of each remaining parameter

Variable	Symbol	Dimension
System	G'	$\text{kg}^0 \cdot \text{s}^2$
Damping	β'	$\text{kg}^0 \cdot \text{s}^{-1}$
Spring constant	k'	$\text{kg}^0 \cdot \text{s}^{-2}$
Laplace variable	s'	s^{-1}

(As before, the Laplace variable is distinguished from the time dimension of seconds by use of italics). Application of the EHF Theorem yields the following relationships between the sensitivities:

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} S_m^G \\ S_\beta^G \\ S_k^G \\ S_s^G \end{bmatrix} \qquad (4.90)$$

And as expected, one of the sensitivity invariant equations has been eliminated by the specially chosen reparameterization.

By the assumptions of Euler's Theorem, we will find that there are d number of invariant sensitivity equations if there are d dimensions. Therefore, repeated application of the above method d times in a row will eliminate all invariant sensitivity subspaces. We can now state the following important result connecting the concepts of sensitivity invariance, dimensional analysis, and the Pi Theorem.

Theorem: *An equation of n parameters spanning d physical dimensions will have d invariant sensitivity subspaces. A reparameterization can be obtained consisting of $n-d$ variables that eliminates the invariant subspaces by careful choice of the dimensional system. The reparameterization that eliminates all sensitivity invariant equations will consist solely of dimensionless arguments.*

The first sentence follows directly from the EHF Theorem. The second sentence follows from the analysis demonstrated above. The third statement is an observation that one dimensional unit space will be lost with each parameterization. A converse statement also follows from the EHF Theorem:

Theorem: *The only system representation that will no invariant sensitivity subspaces will be dimensionless.*

In a classical sensitivity sense, we have unified the results of sensitivity theory, dimensional analysis, and functional invariance. Rather than go through the tedious process of repeatedly parameterizing a problem, we may now directly form a model representation that we know is unique with regard to sensitivity by directly forming a dimensionless representation by any of the many methods presented in this thesis.

4.6 Sensitivity Notions of System Equivalence

In a control theoretic framework, there are many notions of system equivalence. One might say two systems are equivalent if they share the same state-space representations or the same transfer functions. From a sensitivity standpoint, some have argued (namely Perkins and colleagues) for a notion of system equivalence based on nominal equivalence of open-loop and closed-loop outputs. For nonlinear systems, system equivalence notions may be defined locally using sensitivity calculations based on local linearization of the system dynamics; see Frank for details (Frank, 1978).

Any introductory course on linear systems analysis would teach that state-space representations are not unique for a given system and therefore system comparisons (open-loop plant, controller, or closed-loop system) should be made in the transfer function domain or in a canonical state-space form. It should be mentioned that hereafter we implicitly assume that the transfer functions (state-space forms, etc) are of minimum order (i.e. no uncancelled internal dynamics).

The trouble with non-unique state-space representations is usually overcome by the use of transfer-function representations. Such forms can compare both SISO and MIMO system representations, where the MIMO case is formed by creating matrices of transfer functions.

Additionally, local linearization-based comparisons of nonlinear systems can be captured in this framework.

Implicit in any numerical representation of a model is that numerical equivalence implies design equivalence. That is, a controller design for one transfer function representation should work equally well on another system of equivalent numerical coefficients on the transfer function of equivalent form (same number of poles and zeros).

This section argues that such methods of transfer function comparison for equivalence can be inappropriate. This statement applies for transfer function representations, state-space representations, or other differential equation representations including nonlinear forms. Additionally, equivalent measures of system performance may also be inappropriate for comparison, i.e. comparing the output response, cost function-measure, or performance criteria such as infinity norms, rise-times, etc. For simplicity, only the notion of transfer function equivalence is discussed, and the reader may readily extend the arguments of the following section to state-space, time, frequency, or cost-function measures as needed.

Two arguments are presented to demonstrate that traditional transfer functions (state-space representations, differential equations, etc.) are inappropriate objects to judge system equivalency. First, it is argued that all systems with equivalent transfer functions do not generate similar controller designs and therefore the notion of transfer function equivalence is too broad. Next it is argued that systems that do not have equivalent transfer functions may indeed generate similar controller designs (and may in fact be the same system). Therefore, the notion of transfer function equivalence is too narrow.

The first argument is well known, specifically that equivalent transfer functions (frequency responses, etc.) do not imply system equivalence especially with regard to sensitivity (Frank, 1978). The modern study into robustness measures and consequent control design methods is a direct consequence of this observation. The second argument is due to dimensional analysis, and is not usually known or addressed in control theory and it is the point of this thesis to address this issue.

Each of the above arguments is centered on a consideration of system sensitivity and sensitivity invariance, two topics that are central to an effective controller design. A statement of system equivalence is then presented from a dimensionless standpoint that specifically addresses equivalence of relative sensitivity between two systems. It is shown that dimensionless representations resolve many of the issues in comparing different system representations and 'recovers' classical notions of system equivalence in the transfer-function (state-space, freq. response, cost-function, etc.) domains.

4.6.1 Transfer Functions Allow Too Broad a Notion of Equivalence

It is well known that equivalent transfer functions (state-space forms, differential equations, etc) do not imply equivalent system sensitivity. Therefore, the notion of using numerical equivalence of transfer function coefficients (state-space equivalence, differential equation equivalence…) to compare system controller designs is an inappropriate viewpoint. An example from Frank (Frank, 1978) illustrates this point nicely:

Consider a transfer function representation consisting solely of non-dynamic gain elements as shown in the figure below:

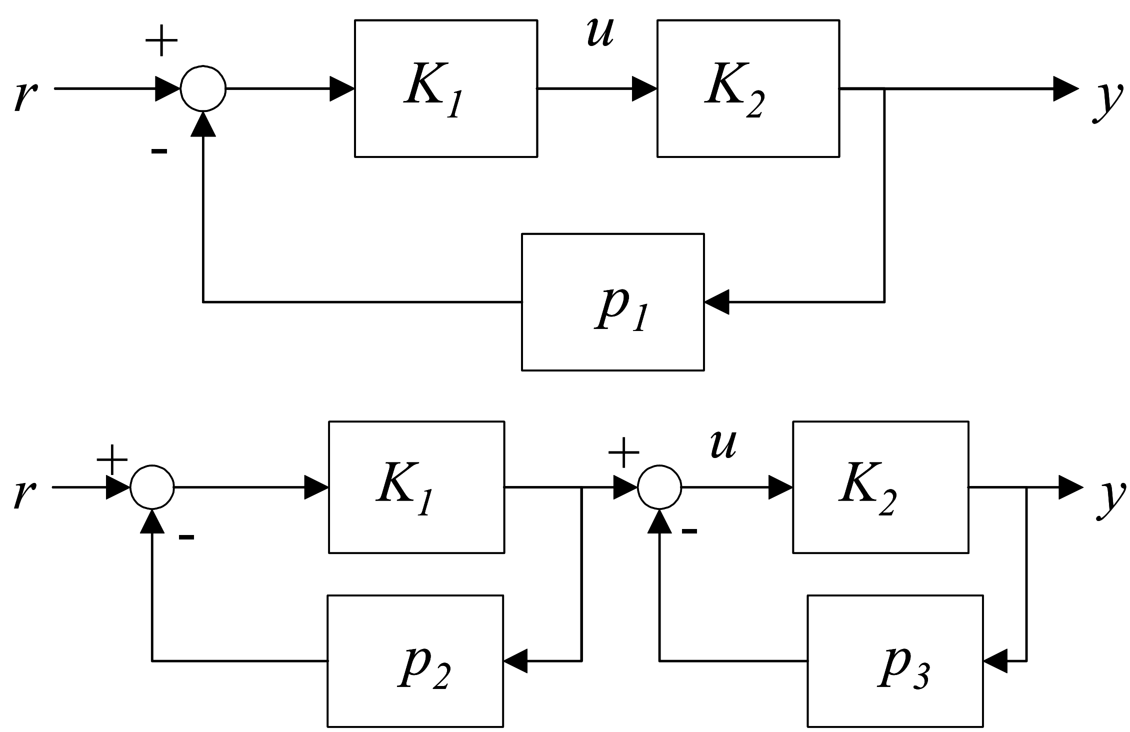


Figure 4.7: Two control loops that are nominally equivalent

If we set $K_1 = 100$, $K_2 = 100$, $p_1 = 0.0099$, $p_2 = 0.09$, and $p_3 = 0.09$, then both control loops have the same nominal transfer function, simply a constant gain value of 100. However, each has different sensitivities with respect to parameter perturbations. For instance, the sensitivity of the top loop with respect to K_2 is given as:

$$S_{K_2}^{G_{top}} = \frac{1 + p_1 \cdot K_1 \cdot K_2}{K_1 \cdot K_2} \cdot K_2 = 0.01 \tag{4.91}$$

Yet the sensitivity of the bottom loop with respect to K_2 is given as:

$$S_{K_2}^{G_{top}} = \frac{1 + p_1 \cdot K_1 \cdot K_2}{K_1 \cdot K_2} \cdot K_2 = 0.01 \tag{4.92}$$

Clearly, the two systems do not have the same parameter sensitivity, and hence a controller design on the first system will not have the same behavior with respect to infinitesimal parameter perturbations. Yet, if one measured the transfer functions (frequency-responses, time responses, etc.), one would NOT be able to discern that the two systems did not share identical sensitivity. A control engineer confronted with the two systems might incorrectly conclude that a design on the first system might generalize to the second system.

In a more embarrassing but realistic example, the author has encountered the above sensitivity problem in practice and it has cost him years of research. During the first years of working on the roadway simulator described in Chapter 2, the author identified the dynamics of the scale vehicle described in Chapter 2 in order to obtain a model for future controller design. The measurements were obtained using standard input/output methods such as swept-sine frequency responses and time-domain measures. The pole locations and transient behavior were found to be identical to those measured on full-sized vehicles, so the model vehicle was thought to be design-equivalent to the full-sized vehicle. A model-reference controller was designed and tested, yet the performance did not change as expected with vehicle speed. At this point, question marks in the identification began to arise. A lengthy component-by-component input/output analysis (and a Master's thesis later) revealed that the pole locations identified were NOT those of the vehicle motion but of the steering actuator, and that the poles of the vehicle were fast enough as to 'disappear' in the roll-off of the actuator dynamics. By nearly pure coincidence, the actuator dynamics matched the behavior of the input/output dynamics of vehicle motion. The controller gain-scheduling with respect to velocity did not work because the scale model was not changing with respect to speed as the full-sized model did; the steering actuator dynamics on the scale-model were constant regardless of vehicle velocity.

As a note, this deception in the identification was not entirely the fault of the author. The steering actuator of the scale vehicle at that time was an R/C servo. A deep discussion with local enthusiasts revealed that R/C servo dynamics are specifically designed to provide a realistic driving or flying experience on a scale-sized R/C car or aircraft. These 'toys' inherently have very fast dynamics due to their large steering-forces and small inertia, so slow steering actuators are designed in a sense to slow the system response by low-pass filtering the input signals. The very bad luck of incorrectly identifying the model is perhaps balanced by the extreme good luck of having a very high local density of R/C enthusiasts not found anywhere else in the world. In fact, Champaign-Urbana is the world headquarters for nearly all of the top R/C companies in the world; Tower Hobbies, Hobbico, Futaba, World Planes, and many others share Champaign as their home. The author has spent many hours in the back-doors of their factories, and the interested reader should refer to the authors Master's thesis for details discovered about R/C systems.

It was because of this failure in identification that dimensional analysis was so vehemently pursued as a necessary approach to system identification and controller design which necessarily lead to the remainder of this thesis. For instance, let us consider a dimensional analysis on the two systems of Figure 4.8. The variables and their dimensions in reference (r), control (u), and output (y) dimension systems are obtained by inspection and are listed as follows:

Variable	Symbol	Dimension
Gain 1	K_1	$r^{-1} \cdot u$
Gain 2	K_2	$u^{-1} \cdot y$
Parameter 1	p_1	$r \cdot y^{-1}$
Parameter 2	p_2	$r \cdot u^{-1}$
Parameter 3	p_3	$u \cdot y^{-1}$

A dimensional analysis shows that the top system in the figure has one pi value:

$$\pi_1 = p_1 \cdot K_1 \cdot K_2 \tag{4.93}$$

While the bottom system has two pi values:

$$\pi_1 = p_2 \cdot K_1, \pi_1 = p_3 \cdot K_2 \tag{4.94}$$

According to the theories of dimensional analysis, no system that is completely described by one pi value can be made dimensionally similar (in the sense of Chapter 3) to a system completely described by two pi values. Dimensional analysis directly predicts that the systems are dissimilar.

4.6.2 Transfer Functions Allow Too Narrow a Notion of Equivalence

Previous arguments and examples showed that transfer function equivalence may be too broad a notion of equivalence. This stems from a rather obvious observation that system sensitivities may not be equivalent between two numerically equal transfer functions. One might conjecture that we may recover a notion of system equivalence by imposing *additional* conditions on our system apart from transfer function equivalence. This section shows that this is conjecture is incorrect and that the notion of transfer function equivalence should be abandoned. This is demonstrated by the fact that two systems may be equivalent and generate equivalent controller designs, but do not satisfy numerical equivalence of transfer functions.

This concept is best illustrated by an example, specifically the mass-spring-damper problem presented in Chapter 1. Recall that the equation of motion was:

$$\frac{Y(s)}{U(s)} = \frac{1}{m \cdot s^2 + \beta \cdot s + k} \tag{4.95}$$

The parameters involved with the system representation are listed below. The asterisk * is to note that an initial energy could be used for systems with zero initial conditions. See discussion on mass spring damper problems referenced in the Appendix of this thesis, specifically the analysis by Kline (Kline, 1965).

With the dimensions of the variables now defined, we can proceed to generate the pi-values associated with the mass-spring-damper using the approach discussed in Chapter 2:

Variable	Symbol	Dimension
characteristic time (rise time, peak time, etc.)	t	s^{-1}
characteristic length (max. amplitude, etc.)	x	m
damping ratio	β	$kg \cdot s^{-1}$
initial position*	x_0	m
spring constant	k	$kg \cdot s^{-2}$
sprung mass	m	kg

The calculation of the pi-values is obtained directly from the following matrix:

$$\begin{array}{c|ccc:ccc} & t & x & \beta & x_0 & k & m \\ \hline \mathrm{m} & 0 & 1 & 0 & 1 & 0 & 0 \\ \mathrm{kg} & 0 & 0 & 1 & 0 & 1 & 1 \\ \mathrm{s} & 1 & 0 & -1 & 0 & -2 & 0 \\ \hdashline \pi_1 & 1 & 0 & 0 & 0 & 1/2 & -1/2 \\ \pi_2 & 0 & 1 & 0 & -1 & 0 & 0 \\ \pi_3 & 0 & 0 & 1 & 0 & -1/2 & -1/2 \end{array} \tag{4.96}$$

The associated pi-values are:

$$\pi_1 = t \cdot \sqrt{\frac{k}{m}}, \pi_2 = \frac{x}{x_0}, \pi_3 = \frac{\beta}{\sqrt{k \cdot m}} \tag{4.97}$$

We now consider the four systems presented earlier and their associated pi-values. Note that the second pi function is not considered, because zero initial conditions are assumed to simulate the step-response (the fact that this pi-value is infinite illustrates the statement mentioned in Chapter 2 that irrelevant pi parameters are generally very large or very small):

System	m [kg]	β [kg/s]	k [kg/s^2]	π_1	π_3
1	1	2	1	t	2
2	10	20	10	t	2
3	0.25	1	1	2t	2
4	1	3	1	3t	3

Because the pi-values of the first two systems are equivalent, both share equivalent relative sensitivity functions. When time is normalized according to π_1, the third system can be made such that the first three systems are equivalent. However, no time or distance normalization can change π_3. By observation, the last system can never be made equivalent with regard to sensitivity.

We now calculate the system sensitivity as a function of time and confirm the above statements. The sensitivities are calculated numerically from:

$$\bar{S}_{\Delta m}^{y}(t) = \frac{\dfrac{Y_{perturb}(t) - Y_0(t)}{Y_0(t)}}{\dfrac{\Delta m}{m}} \tag{4.98}$$

For the following three parameter sets:

(1) $m = 1, \beta = 2,\ k = 1, \Delta m = 0.1$,

(2) $m = 10, \beta = 20,\ k = 10, \Delta m = 1$

(3) $m = 0.25, \beta = 1,\ k = 1, \Delta m = 0.1$

(4) $m = 1, \beta = 3,\ k = 1, \Delta m = 0.1$,

the corresponding nominal (unperturbed) transfer functions are:

$$\frac{Y_1(s)}{U(s)} = \frac{1}{s^2 + 2 \cdot s + 1}, \frac{Y_2(s)}{U(s)} = \frac{1}{10 \cdot s^2 + 20 \cdot s + 10} \tag{4.99}$$

$$\frac{Y_3(s)}{U(s)} = \frac{1}{0.25 \cdot s^2 + s + 1}, \frac{Y_4(s)}{U(s)} = \frac{1}{s^2 + 3 \cdot s + 1} \tag{4.100}$$

Note that the relative parameter change is constant. Using a numerical analysis, plots of the relative sensitivity were made for systems 1 and 2 (left), and systems 1 and 3 (right), and systems 1 and 4 (below) shown in Figure 4.9. If we scale the time by a factor of $\sqrt{\frac{k}{m}}$, we obtain the plots shown in Figure 4.10, the left showing the first three systems and the right showing systems 1 and 4.

Clearly, the first three systems are equivalent with respect to *relative* sensitivity, but not equivalent with respect to their transfer function representations, while the last system cannot be made equivalent to any of the first three systems with regard to sensitivity. Even with only this one example, we can state the following important claim:

Equivalent transfer functions (state-space representations, freq. responses, etc.) do not imply equivalent sensitivity, and equivalent sensitivity does not imply equivalent transfer functions (state-space representations, freq. responses, etc).

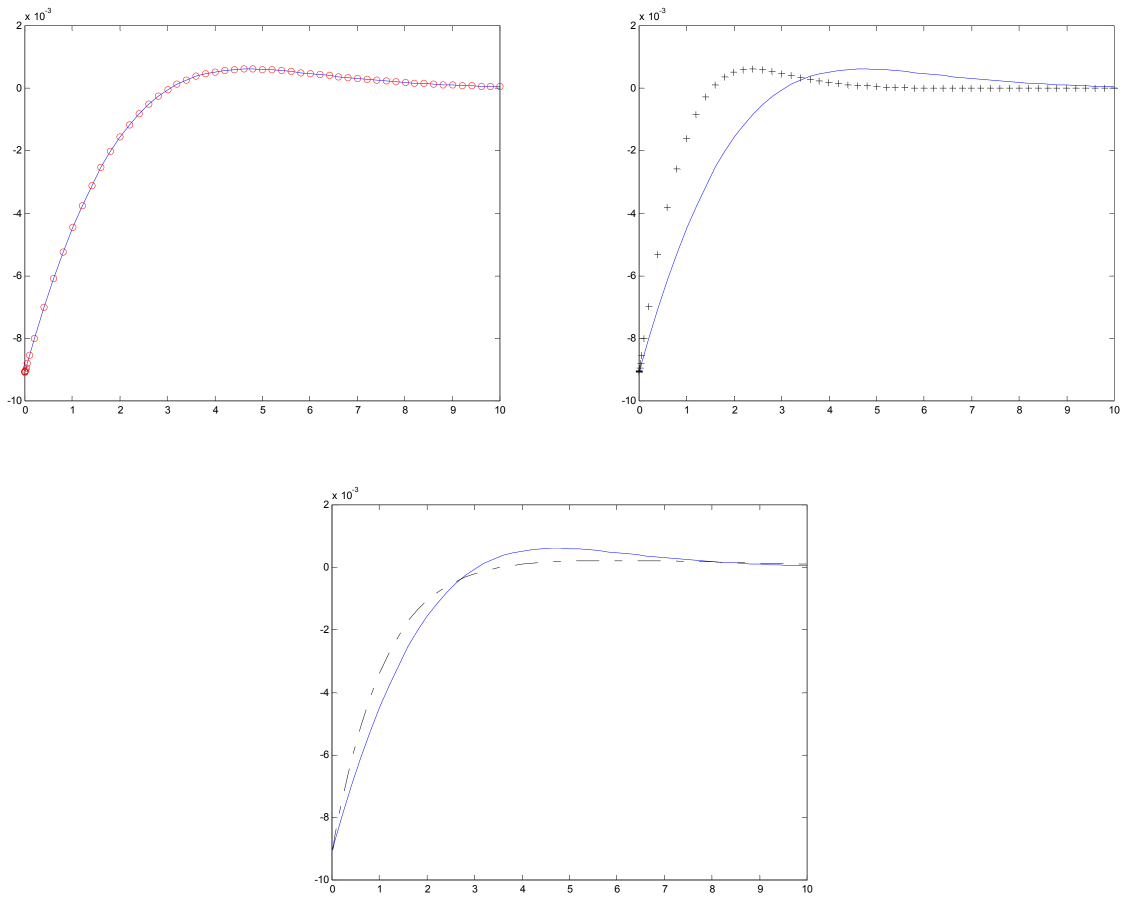


Figure 4.9: Relative sensitivity plots of system 1 vs. 2 (left), vs. 3 (right), and vs. 4 (bottom)

We have argued that standard, numerical measures of systems (i.e. transfer functions, state-space representations, differential equations) misrepresent notions of system equality with regard to controller design. That is, the system sensitivity to variations in parameters, disturbances, or model uncertainty is not well represented by traditional model forms. Therefore, we must question whether a better representation exists that may be more suitable for controller design. Continuing arguments implicit in the presentation above, it is clear that the best system representation for claims of equality is clearly a dimensionless one.

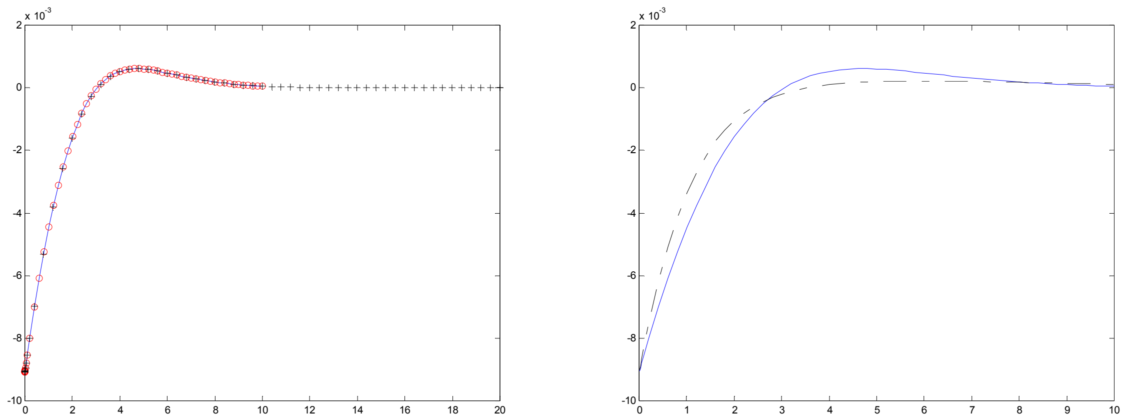


Figure 4.10: Relative sen. of sys. 1-3 (left) and sys. 1 and 4 (right), with time normalized.

4.6.3 Equivalency Based on a Dimensionless Representation of Systems

The previous sections of this chapter hint at a resolution to the conflicting notions of system equivalence. Namely, system-to-system comparisons should incorporate all available information about the model in any equivalence statement. From the arguments must require that the systems under comparison have the following properties:

(1) The systems must have the same invariant sensitivity subspaces and,

(2) The systems must have equivalent numerical values of the model measurements of interest in a dimensionless representation.

The first condition requires that both systems must have the same form of dimensionless pi-parameters. The second condition requires that the numerical value of each pi-parameter is identical in both systems. Both conditions are implicit in the traditional statement that two systems are *dynamically similar* as per the definition of Chapter 3.

From a historical viewpoint, the sensitivity-equivalence based notion of system comparisons agrees well with previous attempts in the controls field to make system-to-system comparisons. For instance, the significant amount of sensitivity work by Horowitz, Perkins and Cruz focused largely on the notion of system equivalence with regard to sensitivity equivalence. Indeed, Perkins and colleagues defined their notion of comparison sensitivity (an extension of Bode sensitivity) as a comparison of system sensitivity between open-loop and closed-loop structures (see the discussion in Frank). This notion of sensitivity is based on a sense of nominal equivalence, where two systems with no perturbations are said to be nominally equivalent if the numerical measurement of their performance are equivalent. The measure may be a state trajectory, an output, a cost function, or a transfer function.

By requiring equivalence of dimensionless model representations, the problems of invariant sensitivities are eliminated. Additionally, a dimensionless notion of system representation has the following additional advantages:

1. Dimensionless representations are simple to obtain and can often be constructed without full knowledge of model dynamics.
2. Dimensionless representations are easy to use. Namely, they allow the use of previously developed control methodologies, design tools, and solutions.
3. Dimensionless representations are well-defined for linear and nonlinear systems.
4. Dimensionless representations have a long history of robust use (albeit, outside the control field).
5. Equivalent dimensionless system representations imply both equivalent dynamics AND equivalent sensitivity to the parameters, disturbances, and unmodeled dynamics explicitly or implicitly represented in the equations or models.
6. Dimensionless representations, like the Laplace approach, extend equally well to systems, signals, or perturbation inputs.
7. Dimensionless representations allow new solution approaches that are not evident in other representations.

Many of the above points are subtle and are argued in more detail in later chapters in this thesis.

4.7 System-to-System Comparisons Using Dimensional Analysis

- The Ox and the Toad -
An ox while drinking by the road
Stepped on the offspring of a toad.
His mother came; he was not there;
She asked her sons if they knew where
He was. "O Mother, he is dead.
We saw a ponderous quadruped
An hour ago who came this way,
Whose hoof quite squashed him where he lay."
The toad then puffed herself all out,
And asked if it had been about
This size. They told their mother, "Stop,
Lest you be torn apart! You'll pop
To bits before you imitate
A creature so immensely great."
- Aesop

Previous discussions (See Chapter 3) have already illustrated that the relevance of various pi-parameters within a system are strongly dependent on many modeling factors, including (1) the desired level of model fidelity and (2) the parameter range and bandwidth over which the model is assumed to be representative. Once the relevant parameters are identified, one of the initial attempts to characterize a system is to compare it to other systems where perhaps a complete analysis has already been completed. In this sense, the notion of system equivalence previously discussed becomes very important, and an obvious criterion for equivalence is that the pi-parameters of the system representations are equivalent.

If the pi-parameters for two different are identical (or very close), and both systems are governed by the same dynamics, then the two systems are traditionally said to be *dynamically similar*. This concept has long been known apart from the sensitivity context just discussed. It is used primarily where is generally used where a full-sized system (called the *prototype*) is compared to a model of the system (called the *model*). If every pi-parameter of the model matches that of the prototype, the two systems are said to be *completely similar*. Note that the model may be larger, smaller, or the same size as the prototype (Langhaar, 1951)(p. 64).

In general, only approximate matching between different systems can be achieved versus exact numerical equality. In such cases, the determination of whether two systems are 'close' with respect to their pi-parameters is determined primarily by the considerations of the experimenter and what aspects of the systems are being compared.

For most analysis (and controller designs), there are more systems under a consideration than a single model and single prototype. Instead, a generalized pi-analysis must be conducted to compare many different members of a group, where each system in the group is assumed to be dynamically similar to every other system in the group. Hereafter, such a group of systems will be referred to as a *system class*, with the usage of the word 'class' consistent with author authors in the field of dimensional analysis (Kline, 1965) (p. 42).

4.7.1 Complete and Partial System Similarity

By analyzing an entire system class, it is often possible to determine a localized operating space in the pi-domain in which the entire class will exist. For instance, comparisons of the pi-parameters governing energy usage for mammals might generate a very small region in the pi-space over which mammals would cluster, perhaps a different region for reptiles, and perhaps another region for insects. In the reverse argument, the analysis of clustering of pi-variables directly defines the notion of a system class, and membership within each class becomes well-defined experimentally. Additionally, measurements of the pi-values of an individual system allow the determination of 'closeness' to defined member sets. This is useful for 'questionable' systems, whose class may not be quite clear. For instance, the pi-parameters of the energy usage of marsupials (for instance, kangaroos) would probably cluster close to the pi-subspace corresponding to a 'mammal', rather than a 'reptile', or 'insect'.

To present the usage in a topic relevant to this thesis, the class of systems studied in this thesis is passenger vehicles at highway speeds. This class may be partially similar to passenger vehicles at low speeds (kinematic relationships), passenger vehicles towing light objects, some types of motorcycles, etc. It would not be similar to aircraft or trains.

4.7.2 Pi Parameters Associated with Optimized Systems

Nearly all mechanical, electrical, network, or biological systems in existence have been optimized to varying degrees by economic, mechanical, power usage, evolutionary selection, etc. Let us consider the general form of these system representations in the dimensionless form to demonstrate that power-law behavior will nearly always be observed in such systems. The power law trends will be a result of the usage of Dimensional Analysis to eliminate the subspaces of invariant behavior discussed earlier. It will also be shown that this optimization generally corresponds to a 'tight' clustering of parameters in pi-space, where each member of a system class generally follows a well-defined distribution about a point center of a cluster.

To demonstrate these properties, consider an arbitrary system representation in the dimensionless domain:

$$\pi_n = f\left(\pi_1, \pi_2, \ldots, \pi_{n-1}\right) \tag{4.101}$$

Let us assume that there are some criteria or cost function on which to evaluate the optimality of the above relationship.

$$\mathrm{C} = f\left(\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n\right) \tag{4.102}$$

This cost function will likely be unknown; for instance, the cost function over which a vehicle design is optimized is dependent on cost, construction, and emotional considerations nearly all of which are impossible to model analytically. Let us assume that the cost function exhibits locations of local optimality given by a locally negative-convex curve. The locally optimal solution will have the well-known gradient property of:

$$\frac{\partial \mathrm{C}}{\partial \pi_1} = 0, \frac{\partial \mathrm{C}}{\partial \pi_2} = 0, \ldots, \frac{\partial \mathrm{C}}{\partial \pi_n} = 0 \tag{4.103}$$

The solution to this optimal problem will be of the form:

$$\pi_1 = \mathrm{c}_1, \pi_2 = \mathrm{c}_2, \ldots, \pi_n = \mathrm{c}_\mathrm{n} \tag{4.104}$$

where the terms $\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_\mathrm{n}$ are numerical constants. In actual physical systems that have been optimized, the pi-values will not all be identical, but they should cluster around some constant location with some type of distribution given by the cost function and their constituent variables will exhibit power laws. These points are discussed in the following sections.

4.7.3 Power-Law Relationships Arising via Dimensional Analysis

For each of the pi variables of Equation 4.104, the pi-parameter relationship can be written in the form:

$$\pi_i = \text{c} \tag{4.105}$$

This will always represent a power-law relationship between its constituent variables. For instance, a constant pi-function can always be written in the form:

$$\pi_i = V_1^a \cdot V_2^b \cdot V_3^c = constant \tag{4.106}$$

where V_1, V_2, V_3 are the variables of the problem used to construct the pi-parameter in earlier reparameterizations. The above equation considers only three variables, but an arbitrary number may be used. Such a function can be rewritten as:

$$V_1 = constant \cdot V_2^{-b/a} \cdot V_3^{-c/a} \cdot \ldots \tag{4.107}$$

Indeed, many researchers finding such power laws in their experimental data claim to find a new 'discovery' or new system property or invariance, when in fact they have often re-discovered a pi-value relationship that has long been known or is easily derived.

A discussion of the engineering, biological, and mathematical mechanisms causing the tendency of optimized systems to tend toward such power law relationships is beyond this thesis. Regardless, the proof-by-examples approach that follows should demonstrate that either human-selection or natural-selection systems will tend toward locally optimized parameter sets. This topic is of primary consideration in the study of quasi-static systems and bifurification theory, and is also examined in the context of feedback-control theory, for instance the recent work of Doyle and others.

In dimensional analysis, a contradiction is sometimes encountered: the governing variables do not appear to form a dimensionless group, but it is clear from the data that a power-law relationship exists *and* that the variables present are sufficient and necessary for the equation description. In such cases, a constant <u>must</u> be introduced that spans the dimensional subspace connecting the two or more variables in the relationship in order to maintain dimensional homogeneity. Finding such constants is troublesome, as they may seem to arbitrary appear and thus dimensionally connect variables that otherwise could not appear in a significant pi-group. A good example is the constant *G* in the law of universal gravitation, a constant that connects mass, time, and distance. Discovered by Kepler, this constant was inferred by an observed power-law relationship of the planets (Kepler's Third Law) and had to be introduced into his equation to maintain dimensional homogeneity. These constants are called *physical constants* in the literature, and their appearance in physical equations has caused significant debate. Some authors, notably Bridgman (Bridgman, 1943), are of the opinion that such constants should be avoided *unless absolutely necessary.* Others, such as Sedov (Sedov, 1959), state that physical constants must be included whenever they are 'essential' and should not be avoided. Neither statement is useful to an experimentalist in the study of a new phenomenon, since such constants are generally unknown until *after* significant system understanding is already obtained.

If history is a judge, then the natural method of research is to first establish the physical constants by searching first for power-laws between dimensionally decoupled variables. After these constants are found, then the underlying governing law may be found by dimensional analysis. Especially in the study of new fields, the appearance of power laws and hence new physical constants is almost ubiquitous. This in turn hints that some underlying governing law or relationship exists that is unknown. For instance, had Kepler not already found the dimensional constant G, Newton might have had a much more challenging problem in framing a universal law of gravitation.

Power laws in data analysis can therefore be produced by two mechanisms, by pi-constraints or by dimensional constants. A power law of the first type is a statement of *existing* pi-values and it is often not useful other than to obtain and present data to be used to establish governing dynamics. Data presented in this manner might is *always* better presented in a dimensionless manner (pi-pi plots, relative histograms, etc). A power law of the second type is an investigative means to create *new* pi-values, since the discovery of a new physical constant therefore implies a new physical law, and hence many new pi-parameters to be created. Examples are easily found in literature, some presented below, that show both power laws of both types. Unfortunately, it is often unclear in publications which of the two scenarios are being investigated. Hence, there is a significant amount of confusion among many who believe they have discovered new physical laws when they indeed have discovered an invariant sensitivity subspace predicted by dimensional analysis.

4.7.3.1 Mechanical Systems

Mechanical systems are in general optimized with respect to many different design parameters simultaneously, and in most cases the cost criterion is straightforward: minimize the financial cost of a device but maintain the usability from an engineering standpoint. Such an optimization is evident in even the most basic engineering objects, and the nail example from McMahon (McMahon and Bonner, 1983) serves to illustrate this point nicely. Shown below is a plot of nail-length versus diameter.

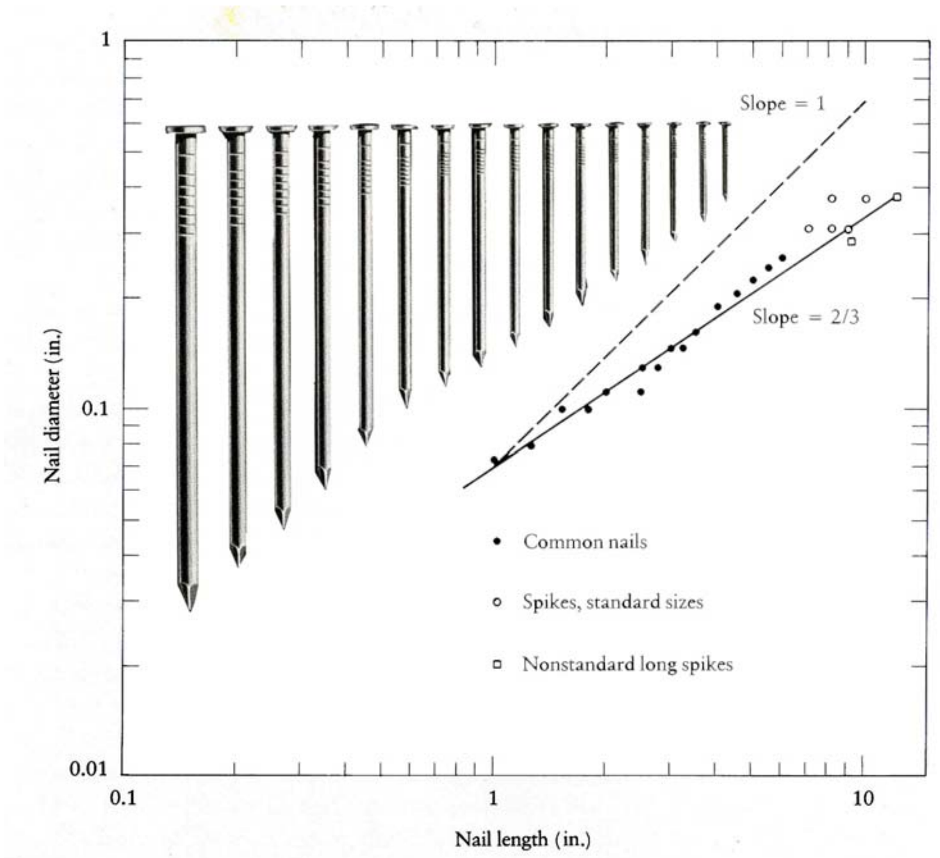


Figure 4.11: Shapes of nails showing power-law relationship

(McMahon and Bonner, 1983)

There are several points to note about this plot:

(1) Nails follow a size-diameter relationship that clearly exhibits a simple power-law relationship

(2) The power-law relationship is not based on maintaining allometric similarity (exact ratios of proportion), as evidenced by the fact that the data do not fit a 1:1 ratio plot shown above as a slope 1 line.

(3) The power-law relationship seems to fit a line of power 2/3.

One might question where the 2/3 power arises. The engineering answer to this question is somewhat obvious, as the size of the nail is generally made to be the minimum diameter to generally prevent buckling in the nail during the impact of the nail with the hammer. According to bucking theory, the diameter of a nail needed to prevent buckling in a beam for a given load grows as a 2/3 power of the length of the beam. The discrepancy of the nail diameters at larger lengths is due to the larger factor of safety needed to prevent buckling due to the use of larger and bulkier hammers that are often used with these nails. From a sensitivity viewpoint, there is an invariance equation imposed on the nail 'model' by physical design criteria in addition to simple dimensional scaling: the law of beam-buckling.

If nails represent one of the simplest mechanical objects in common use, then internal combustion engines represent perhaps the opposite extreme in mechanical complexity. Internal combustion engines also exhibit power-law relationships. The figures below present data also published in McMahon and Bonner (McMahon and Bonner, 1983). The governing law for IC engines is assumed to be unknown, so the following plots are investigating physical constants. The plot below reveals that Base Horse Power (BHP) and Engine Mass are linearly related,

$$HP = k_1 \cdot m \qquad (4.108)$$

and that RPM (at peak horsepower) and engine mass are related by:

$$RPM = k_2 \cdot m^{-1/3} \qquad (4.109)$$

where k_1 and k_2 are physical constants of unknown theoretical origin. The intersection of the plots holds no special meaning, as the two plots are shown on the same graph for convenience and the intersection can be moved arbitrarily by changing the scale of the axes.

The plot shown in Figure 4.13 (from the same source) might be thought of as a pi-relationship if it was assumed that volume scales as the cube of length. However, this is not the case, since the plot clearly does not follow a cubic-power relationship. The *deviation* away from this expected relationship is the point of the plot, as it illustrates that engines, as air-pumps, are designed under constraints other than volumetric considerations.

It is not difficult to determine the underlying physical laws that generate the physical constants; the limiting dynamic in the design of internal combustion engines is that the volumetric rate of air that can pass through the engine is limited by the speed of sound (which limits the rate at which air can enter the cylinder). Additionally, the size is limited by the maximum stresses on the crank arms (which limits rotational rates due to dynamic stresses).

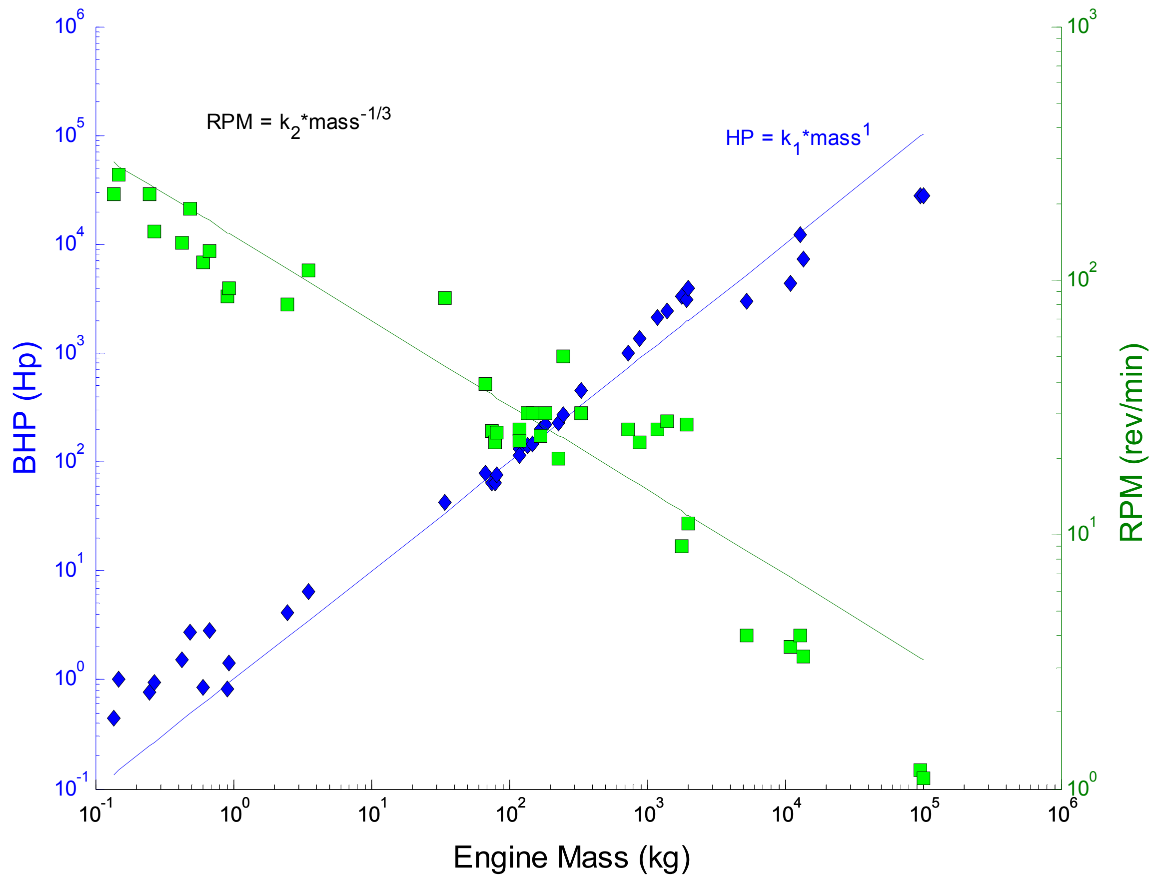


Figure 4.12: Similarity relations for IC engines: horsepower and RPM vs. mass
(McMahon and Bonner, 1983)

The engine plots are quite valuable in terms of methods of designing engines (and aircraft that use them). For a complete dimensional analysis of IC engines, see the above reference or the discussion in Taylor on IC engines (references are provided in the Appendix).

4.7.3.2 Biological Systems

In many respects, biological systems are not as well understood as mechanical or electrical systems simply because of the underlying complexity. However, there has been recent efforts in the last century to discover biological 'laws' of design. It is clear that such laws must exist, as there is a multitude of scaling laws (and hence dimensional laws) known to exist for most biological systems. An excellent reference for biological scaling material, indeed one of

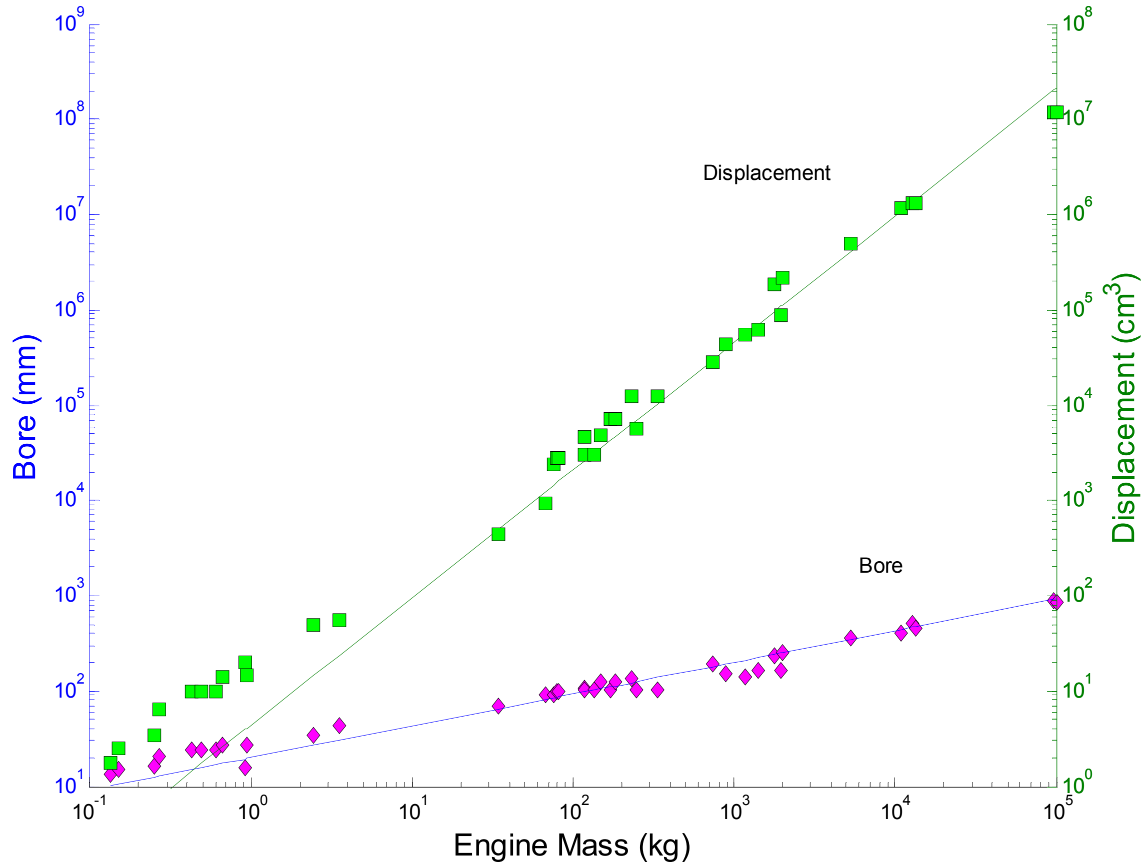


Figure 4.13: Similarity relations for IC engines: displacement and bore vs. mass

(McMahon and Bonner, 1983)

the best reading texts on dimensional analysis, is the book *On Size and Life* by McMahon and Bonner and published by the Scientific American Press (McMahon and Bonner, 1983).

Some simple allometric relationships are exhibited in the following figures, and primarily reveal a constancy of density and similarity in construction. Namely, mass scales with length cubed and surface area scales as length squared. This allometric relationship is seen in the plots below (from McMahon):

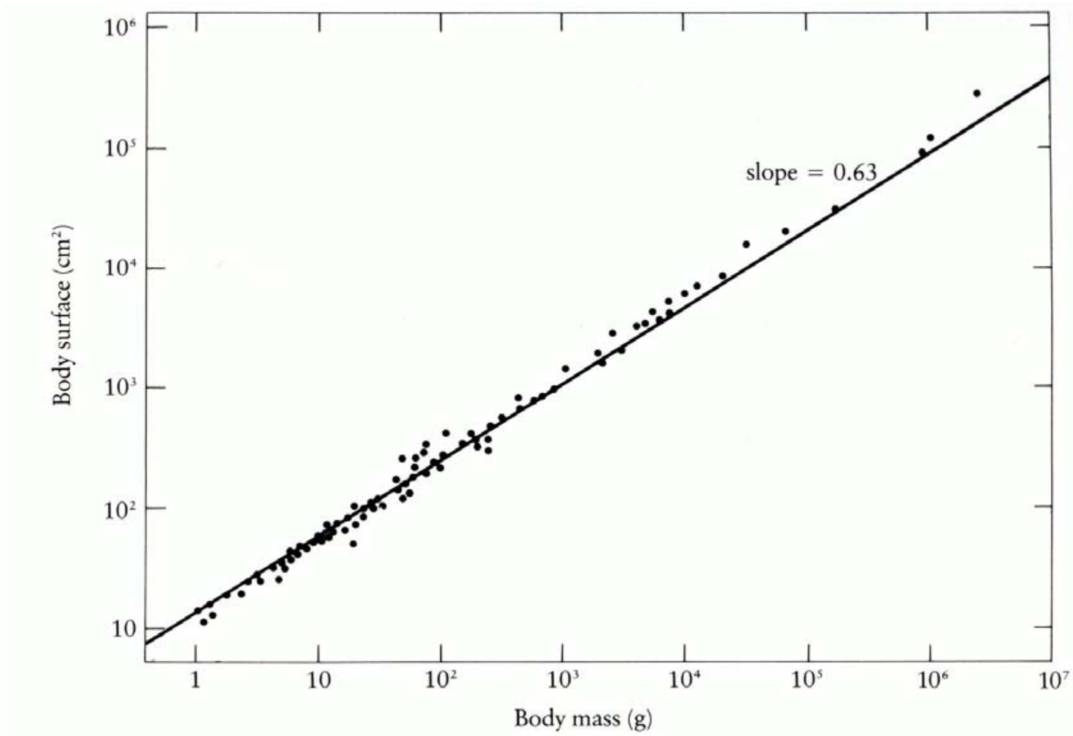


Figure 4.14: Mass versus surface area for a variety of invertebrates

(McMahon and Bonner, 1983)

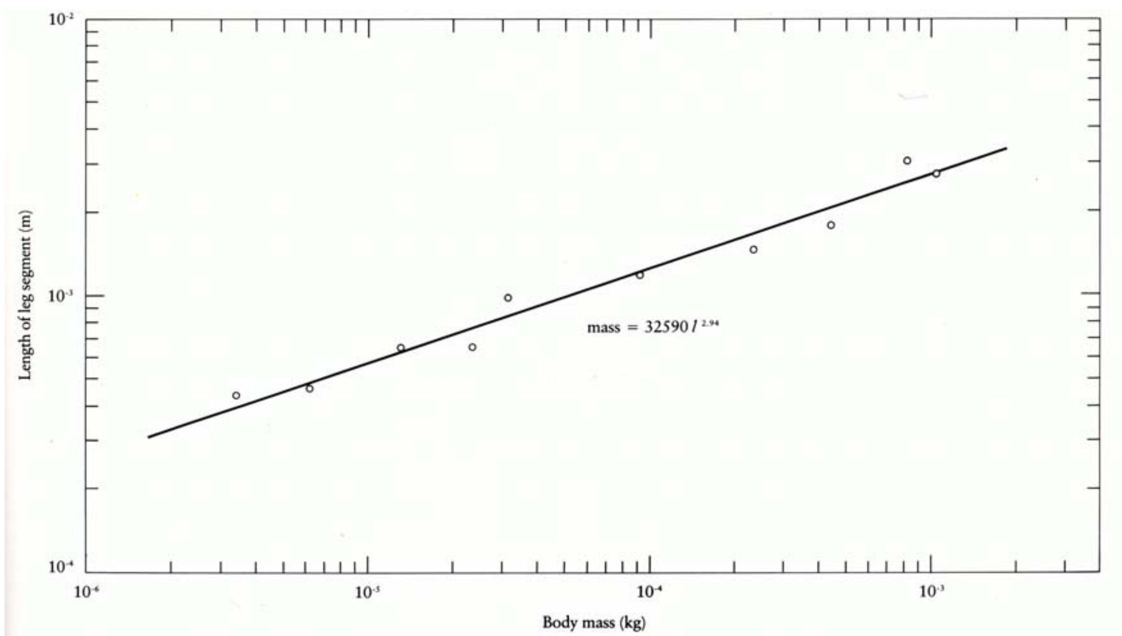


Figure 4.15: Mass versus length for a species of insect

(McMahon and Bonner, 1983)

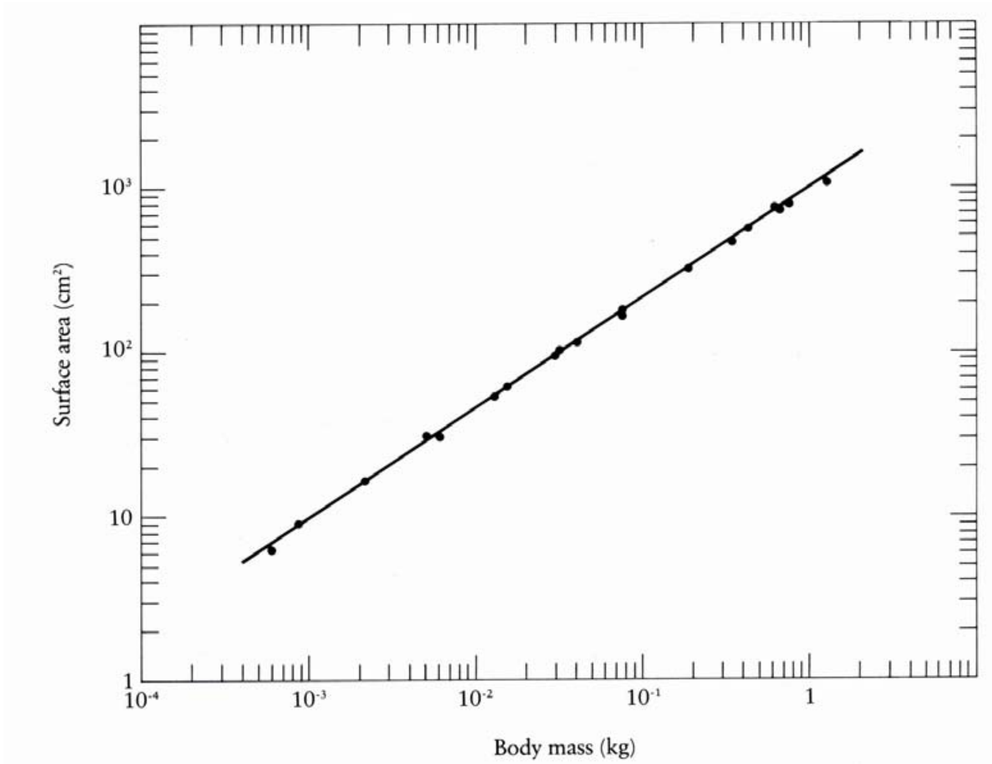


Figure 4.16: Mass versus surface area for a species of salamander

(McMahon and Bonner, 1983)

As an example of a non-allometric invariance present in biological systems, we might investigate the relationship between muscle strength and the mass of the muscle. Because the muscle strength scales as the number of fibers in the muscle, we may conclude that muscle strength scales as length cubed, where length is some characteristic length of the organism. The mass of the organism scales as length cubed, assuming similar density distributions between organisms of different sizes. We therefore conclude that an organism of given weight can lift objects in proportion to it's own weight to the power of 2/3. This is seen quite dramatically in the weight lifting records for humans, shown in the figure below. The records were recorded as sum of press, snatch, and clean-and-jerk versus body weight (all in lbs) (McMahon and Bonner, 1983). One might think of this relationship as a sensitivity invariance relationship between mass of an organism and available force from an organism.

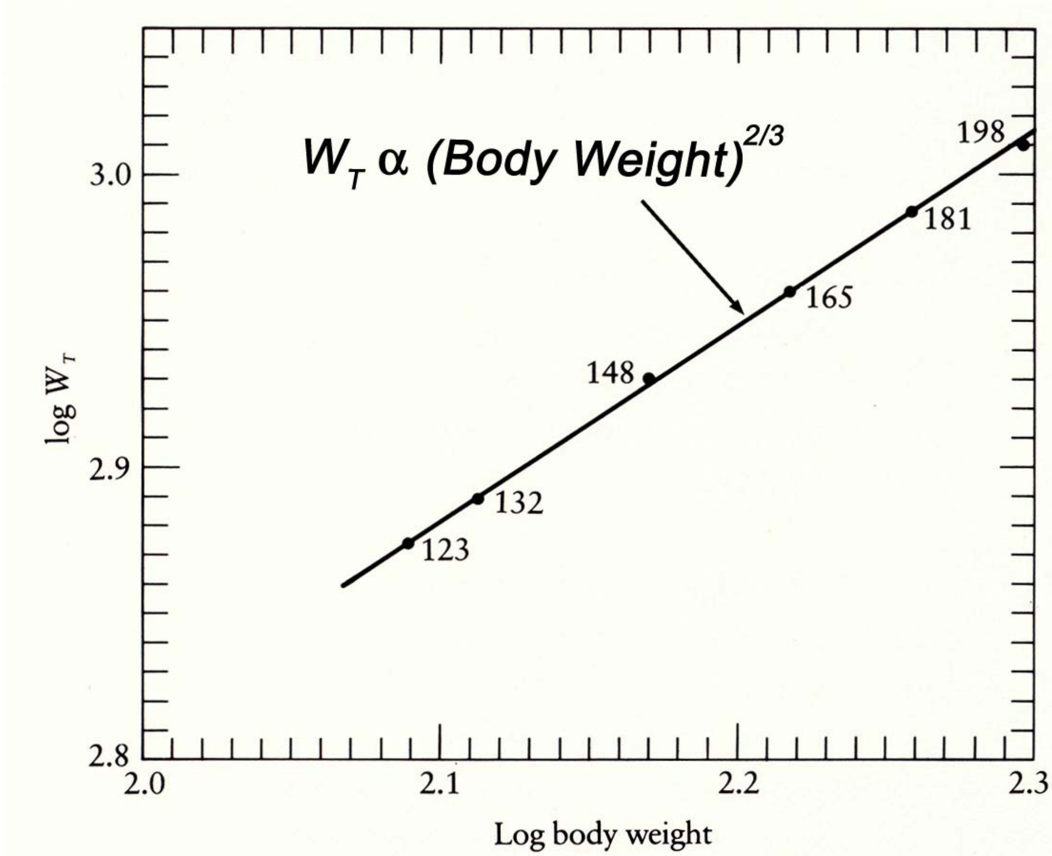


Figure 4.17: World weight lifting records

(McMahon and Bonner, 1983)

The above example hints that muscle fiber tends to maintain a constant stress in each fiber. Let us extend this heuristic reasoning to the heart muscle, where we have a cavity A_1 in the figure below surrounded by muscle of cross-sectional area A_2. By Newton's laws, the force inside the heart tending toward outward expansion, namely the internal pressure times the area A_1, must be equal to the muscle force pulling inward, namely the muscle stress times the area, A_2. During contraction, the muscle stress remains constant, so the ejection pressure on average must be constant for the heart, regardless of size. This is shown in the figure below. Again, the constant muscle-stress relationship may be inferred as an invariant sensitivity relation between different muscles.

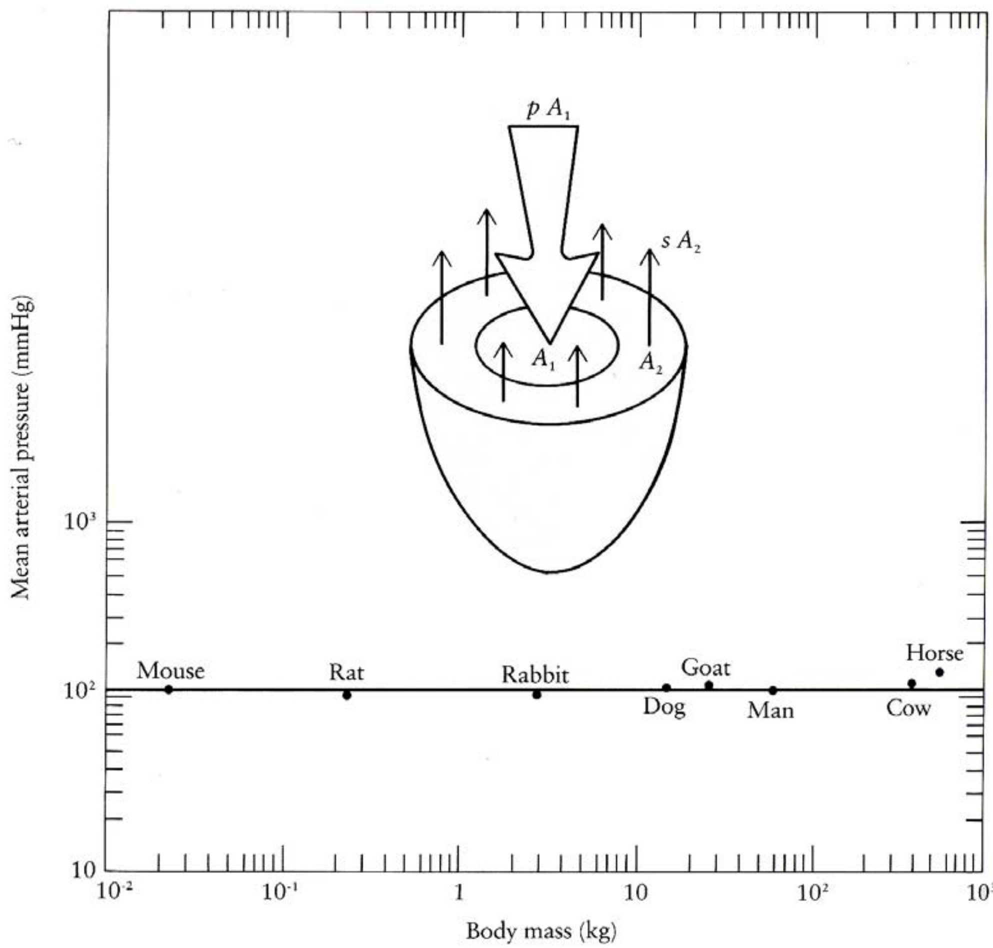


Figure 4.18: Ventricular ejection pressure does not depend on body mass or size

(McMahon and Bonner, 1983)

Related to this problem we may find invariance relationships in the performance of animals. For instance, let us assume that many different, similarly constructed animals jump from resting position straight into the air. We would like to determine the maximum height that such animals can jump. The relevant variables are assumed to be:

Variable	**Symbol**	**Dimension**
jumping height	h	m
mass of animal	m	kg
energy per mass of muscle	Q	$m^2 \cdot s^{-2}$
gravitational acceleration	g	m^2

We wish to find an equation for the maximum height. By the Pi Theorem of Chapter 3, there are $N_P = N_V - N_d = 4 - 3 = 1$ independent dimensionless variable groupings. The dimensional set is therefore given as:

	h	m	Q	g
m	1	0	2	1
kg	0	1	0	0
s	0	0	−2	−2
π_1	1	0	−1	1

(4.110)

Clearly, mass is an irrelevant variable! The dimensionless grouping can be read directly from the bottom row:

$$\pi_1 = \frac{h \cdot g}{Q} = constant \qquad (4.111)$$

Note that the energy content of muscle tissue across all species (even most insects) is constant, so the jumping height of all species is predicted to be approximately constant. Indeed across all mammals (man included), reptiles, birds, and insects, the maximum jumping height for each is approximately 2.5 meters if aerodynamic effects are ignored. This problem indicates an invariance of the power-density of muscle tissue between different organisms.

If one considers general methods of locomotion, one finds that the size of animals scales with a relationship to mass known as Kleiber's law. This law is derived by noting that locomotion of animals can be modeled as one of two ways: an inverted pendulum or as an elastic spring-motion. For simple walking, the pendulum model is appropriate, but for running the elastic model is most appropriate. Each mode predicts a different power-law between speed (pace) and animal size, and the fact that animals are sized with respect to running should not be a surprise. If one were a predator or prey and could choose whether to optimize for walking or running, the choice of which aspect to optimize for would be obvious. The option of catching dinner (or not being dinner) is certainly more preferable to saving energy in walking long distances. Kleiber's law coupled with a constant energy usage per unit muscle generates power laws of the energy to the power of 0.75 times the length, which is a good fit to predicting heat production for an animal of given mass (also shown are the allometric scaling laws: energy to the 2/3 power of length).

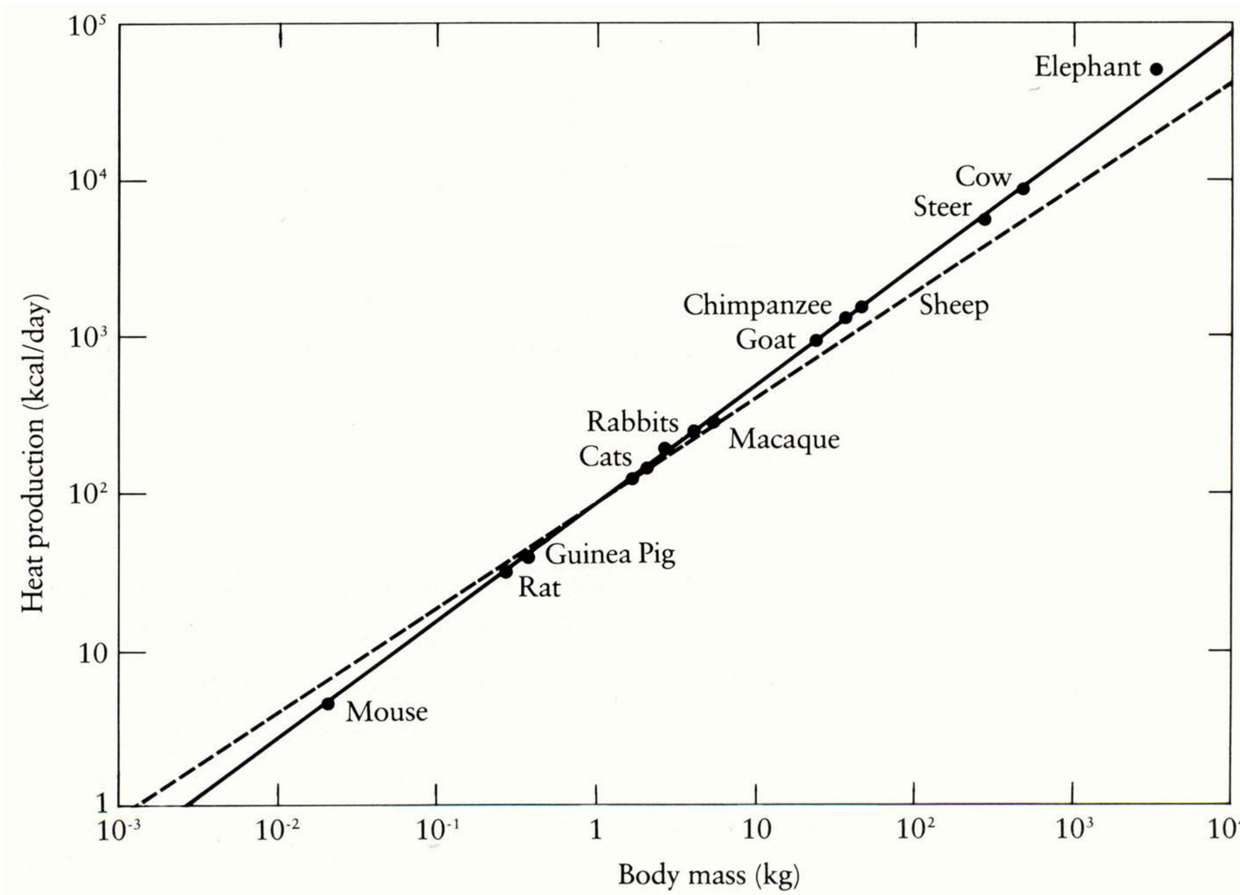


Figure 4.19: Kleiber's law used to predict heat production per mass for a range of masses

(McMahon and Bonner, 1983)

One might extrapolate the above invariance relationship of biological energy to a system-energy balance on a biological scale, where the sun provides a constant unit of energy per surface area. If one assumes a given efficiency of energy utilization for a given length, then one can combine Klieber's law to obtain a population-density estimate per unit surface area of a given biome. Such relationships have been measured, and one is shown below:

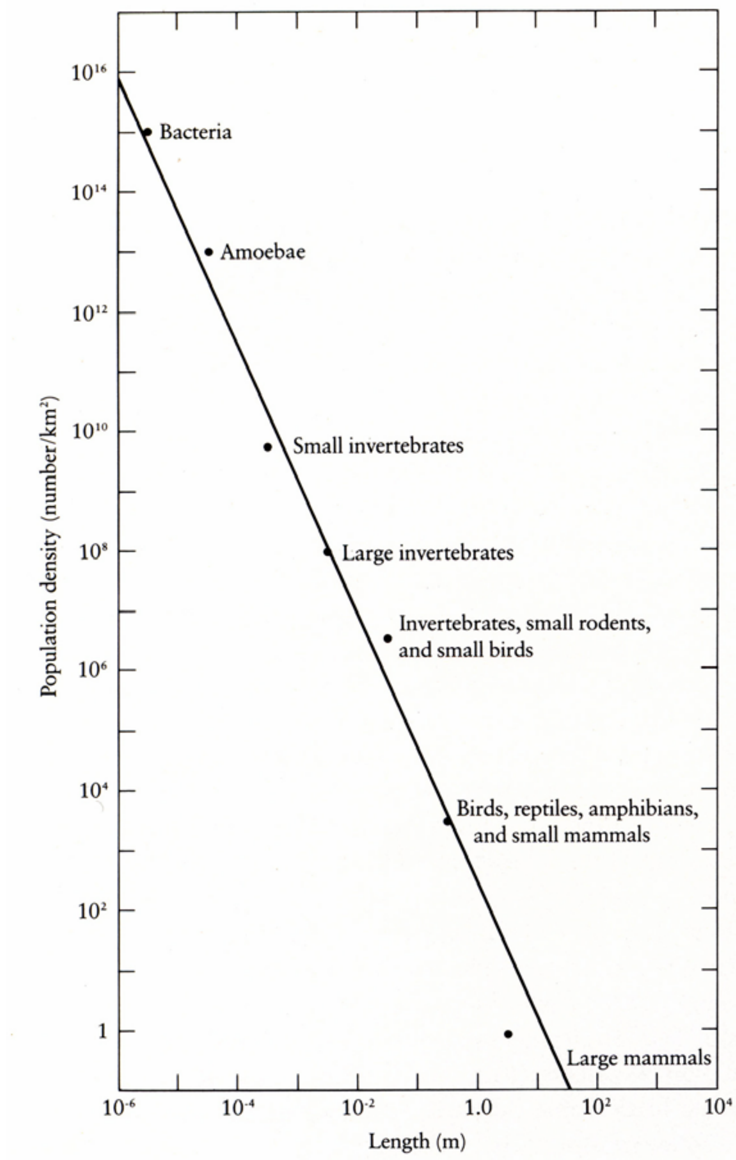


Figure 4.20: Population density versus length, scaling at -2.25 power

(McMahon and Bonner, 1983)

One can see from the above figure that it is highly unlikely that a kilometer-scale animal will exist on the planet.

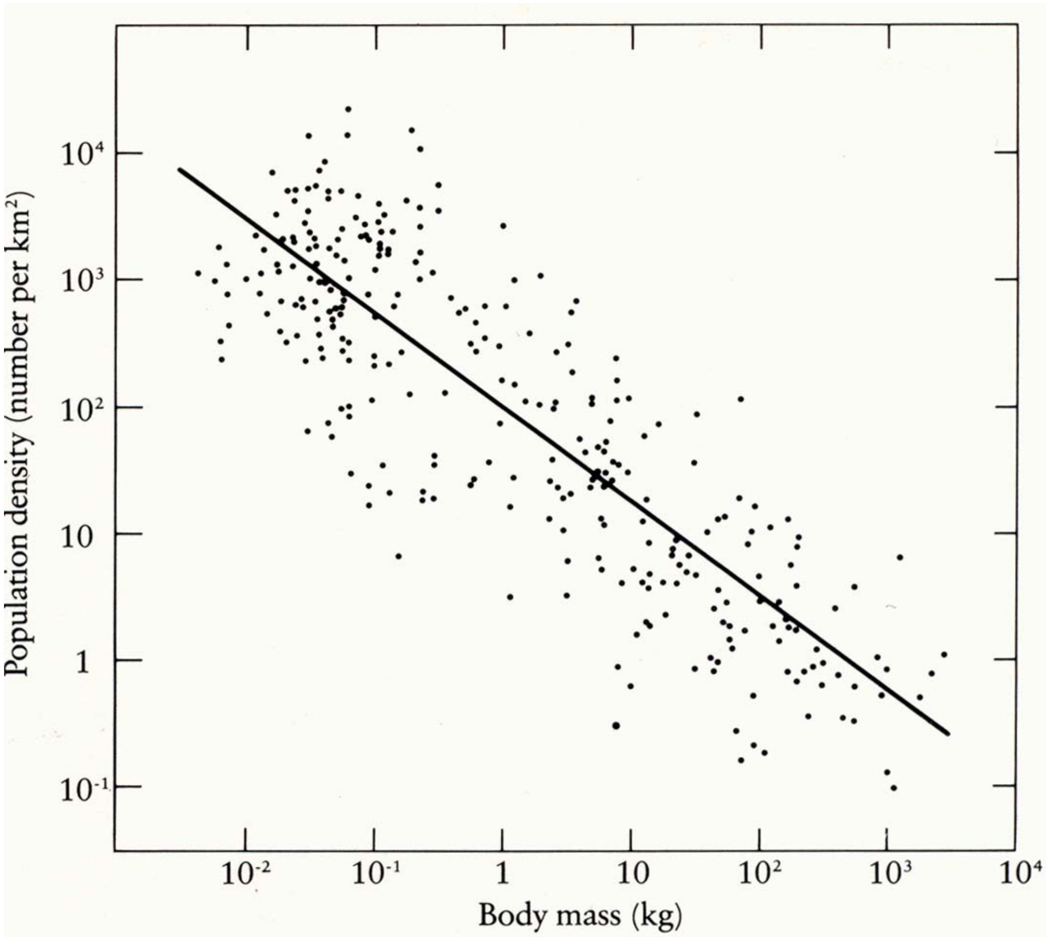


Figure 4.21: Population density versus mass, scaling as -0.75 power

(McMahon and Bonner, 1983)

One might wonder where humans lie on the plots of Figure 4.20 or 4.21. If the above plots are analyzed at a body mass of ~100 kg, then the predicted population density *possible* for man on the planet (based on the surface area of dry, non-polar land) is 1 billion plus or minus an order of magnitude. The fact that mankind has recently been able to exceed this population in the last century (and will hit 10 billion in the author's lifetime) is a statement to our own energy balance. This growth relies on a strong dependence on fossil fuels. Scaling relationships show that a dependence on solar energy with vegetation as the solar collector must impose upper limits to the feasible population of the planet that humans are already exceeding or are rapidly approaching.

To complete the discussion of biological scaling laws, the notion of population density restricted by an energy balance is then used to predict a 'species density'. To do this, it is simple to assume that the number of potential species of a given size must be proportional to the number of individuals of a given size. This generates an extrapolation of species shown below:

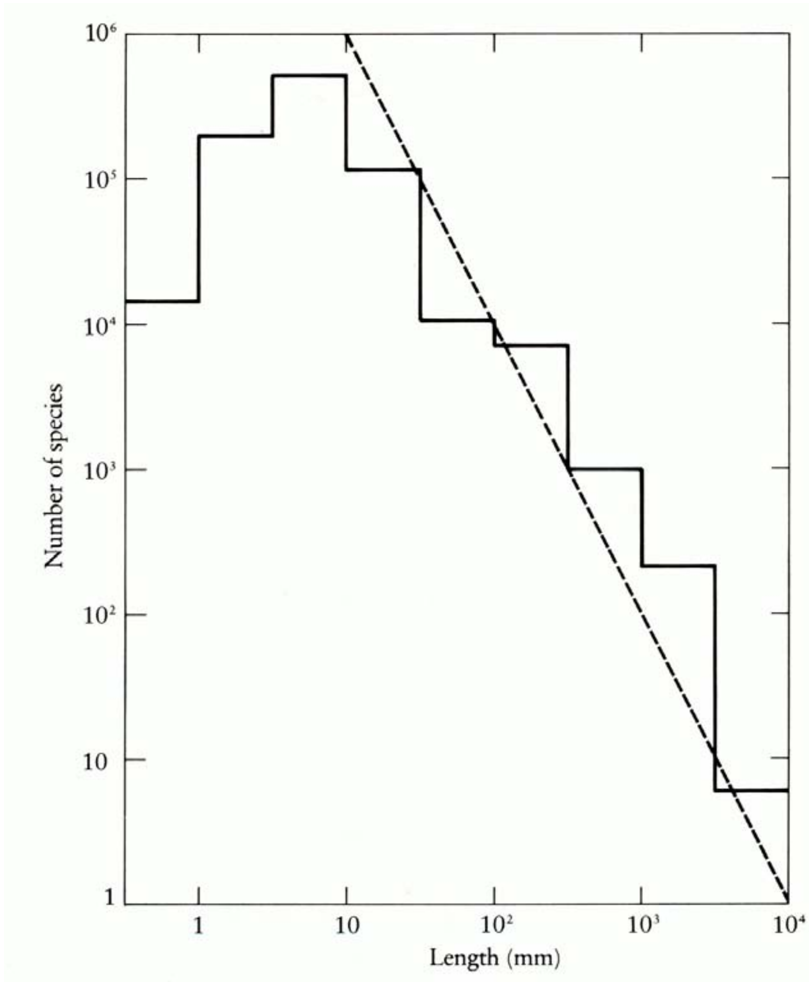


Figure 4.22: Species density versus size

(McMahon and Bonner, 1983)

The above plot hints that there is a vast majority of species that remain undiscovered at very small size scales. Indeed, since this plot was published (mid 1970's), an entire new kingdom of microorganisms has been found and the assumed mass of single-celled bacteria and microorganisms is today thought to be many times greater than all other organisms combined.

4.7.4 Pi-Parameter Distributions Arising via Dimensional Analysis

In an application more relevant to the primary, vehicle-control focus of the thesis, the scaling laws of vehicle design are now considered. Rather than present the data as power-law relationships (which require description of each power-law), distributions of pi-parameters are presented. If the distribution is 'tight' then this corresponds to a 'good' clustering of measurements about a power-law given by the associated pi-parameter. This allows a more compact presentation and simplified analysis for later controller designs.

4.7.4.1 Case Study: Vehicle Chassis Design

The previous discussion noted that a dimensionless system representation, a local pi-parameter optimization should correspond to tight clustering of parameters in the pi-space. Some notion of cost-function smoothness is implied in the cost-function gradient optimization, and therefore we might expect that relative distributions of optimized parameters should exhibit well-defined distributions that are also smooth.

The ideas presented in below on optimized parameter distributions originated from the study of vehicle dynamics discussed in Chapter 2. Because the test vehicles used are a fraction of the size of production vehicles, significant consideration must be given to the design of the test vehicle to ensure a match to full-sized vehicle dynamics. The obvious goal of sizing the test vehicle is to make performance characteristics as 'average' as possible, yet the concept of an 'average' full-sized vehicle dynamic is dimensionally unclear for clear reasons discussed shortly. However, previous discussions have alluded to a tendency of pi-values to tend toward constant values, especially for highly optimized systems. Therefore, we compare the distribution of pi values to that of traditional, dimensioned values as a means of determining an average design. Shown in Figure 4.23 are relative distributions of the moment of inertia and the dimensionless counterpart for production vehicles (the sources of these data is provided in the appendix). The values in red were obtained from the NHTSA database, the values in blue from vehicle dynamics publications in the literature (listed in Appendix C). The plots of Figure 4.23 provide an interesting comparison, namely that the dimensioned parameter distribution is skewed, while the dimensionless parameter exhibits a distribution that does not appear as skewed.

Another interesting aspect of the distributions that deserves discussion is that a large portion (~15%) of 80 vehicle parameters found in journal and conference publications are outliers in the pi-domain (in the sense that they are more than 3 standard-deviations from the

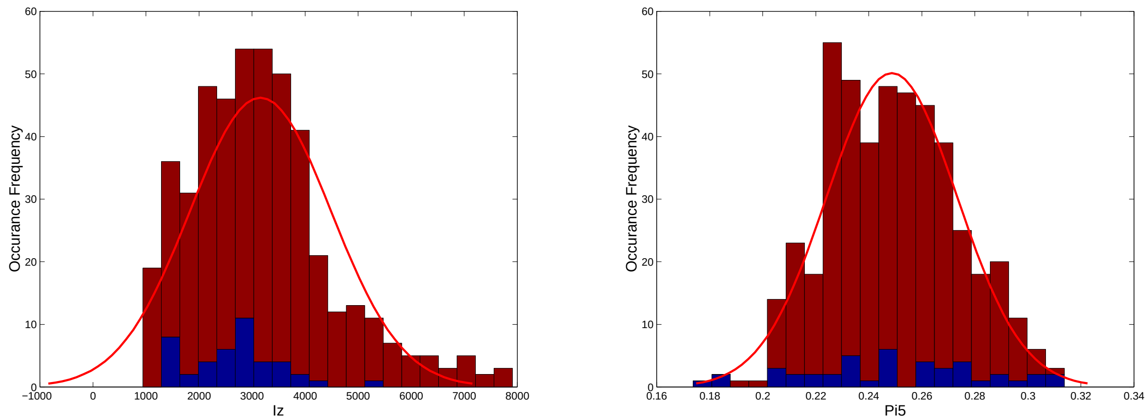


Figure 4.23: Dimensional and dimensionless distributions of the same parameter

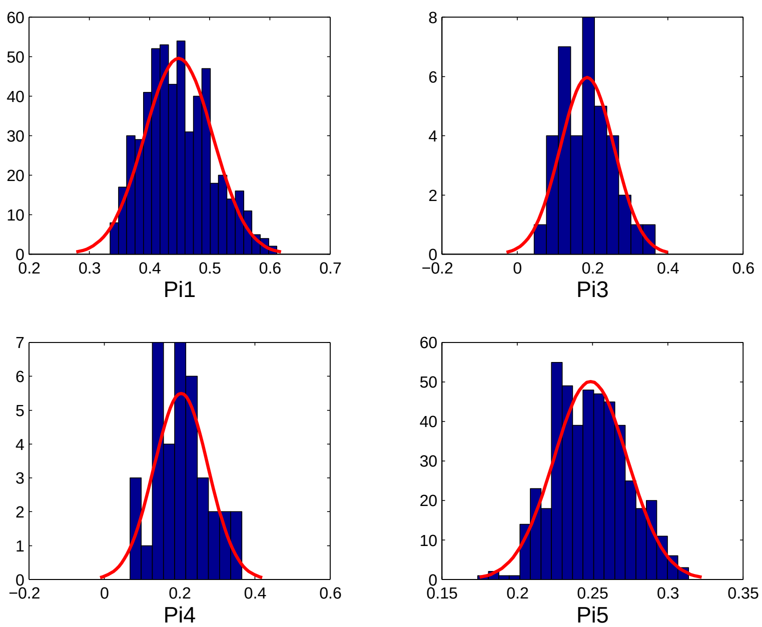


Figure 4.24: Distribution of Pi parameters

average), yet not a single vehicle of the 700 vehicles tested in the NHTSA database appear as outliers in the pi-domain. These very strong outliers in the vehicle-control community are not obvious during presentation, as the parameters appear to be valid in the traditional, dimensioned domain. One must question how they arose, and it is surmised that these parameters were not measured. Instead, they were likely 'formed' by independent averaging of the parameters of several vehicles, a process that will be shown to be inappropriate because it destroys sensitivity invariant subspaces. This is the topic of the following section.

4.7.4.2 Definition of a Nominal System in a Class

A well-defined parameter distribution inherently provides a very meaningful measure of the average and deviation of system behavior from the average. In this sense, the use of pi-parameters of dimensional parameters is already advantageous. However, there are additional reasons to choose a dimensionless parameter representation to define an average system behavior.

To define the average (or other characteristic) of a set of systems, care must be exercised not to destroy inherent relationships given by sensitivity invariants. An example very clearly illustrates why this is the case. Consider *N* cubes of different lengths but constant densities.

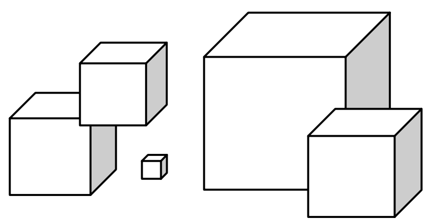


Figure 4.25: A system class of geometrically similar, homogenous cubes

The mass of similar objects will scale as the length of the objects cubed, represented functionally by:

$$m_i = k \cdot x_i^3 \qquad (4.112)$$

where k is the density. Now if we determine an average length for the set of objects, cube this average and multiply it by the density, it is clearly not equal to the average mass of the set of objects. Mathematically:

$$k \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} x_i \right)^3 \neq \frac{1}{N} \cdot \sum_{i=1}^{N} \left(k \cdot x_i^3 \right) \qquad (4.113)$$

If one separately averaged the masses and lengths, and then calculated the average density, the value obtained would not be the density of any of the blocks, which is a constant for all blocks.

Thus, this method of averaging parameters independently would create a system that has an incorrect physical meaning. Because physical parameters are indeed coupled dimensionally, independent averaging of physical parameters to obtain an average system is clearly inappropriate. When even more physical parameters are present, and the dimensional nature of parameters may involve disparate dimensions such as areas, rates, velocities, densities, masses, and forces. In this case, there will be a very large amount of physical confounding of the variables and the potential for subtle dimensional errors due to destruction of sensitivity invariants increases.

4.7.4.3 Example: The Definition of a Nominal Vehicle Dynamic

For the control-law designer, the obvious relevance of the previous pi-distributions is to provide a measure of a system independent of dimensions. The vehicle distributions directly reveal the mean pi-values that therefore define an 'average' vehicle, and because these parameters are dimensionless, they are not physically interdependent based on sizing laws as was seen in the cubes-example.

As an example, the average values for the vehicle pi-distributions presented in the earlier figures are as follows:

$$\overline{\pi}_1 = 0.4431, \overline{\pi}_3 = \frac{145.6771}{\mathrm{U}^2}, \overline{\pi}_4 = 1.0977 \cdot \overline{\pi}_3, \overline{\pi}_5 = 0.2510 \qquad (4.114)$$

The above relationship, as a function of velocity, can then be used to define a nominal system performance. This is obviously useful for robust vehicle controller designs, two of which are illustrated at the end of Chapters 5 and 6.

4.8 Contributions of This Chapter

The primary contributions of this chapter are as follows, numbered by relation to corresponding sections of the chapter:

(1) The discussion of parametric uncertainty in the remainder of the chapter (and thesis) shares analysis techniques and results with the analysis of system behavior in the presence of disturbances. In many (perhaps most) circumstances, one can represent

parametric variations with exogenous disturbances, or in the reverse argument, one may often represent true disturbances via exogenous parametric variations.

(2) There is a rich history and field of study related to sensitivity analysis with strong ties to the University of Illinois. The historical sensitivity notion of the Miller-Murray classification system delimits major areas of focus within each chapter in this thesis.

(3) Euler's Homogenous Function Theorem (EHF Theorem) was used to prove the Pi-Theorem of the last chapter. This theorem predicts equations for invariant Bode sensitivity for a problem representation. Specifically:

  a. Every dimensioned system representation contains at least one and usually multiple subspaces of sensitivity invariance.
  b. The sensitivity invariance is always described by a set of linear equations.
  c. For a system of n parameters spanning d physical dimensions, there will be (n-d) equations of sensitivity invariance.

(4) Examples illustrate that:

  a. System Bode sensitivities to parameter perturbations are usually coupled to sensitivity to state variables (i.e. state-derivatives) by sensitivity invariance equations.
  b. The sensitivity of mappings (equations) consisting only of one dimensionless parameter are often determinable without knowledge of the system dynamics.
  c. The sensitivity invariants apply in static mappings, time-domain dynamic representations, or frequency-domain representations of systems in open or closed feedback loops.
  d. Numerical or algebraic calculation of the sensitivities can be used to verify invariance relationships

(5) The equations for sensitivity invariants can be eliminated by a reparameterization of the system equations using parameter ratios.

  a. Each system reparameterization to eliminate one sensitivity invariance equation eliminates one dimensioned parameter from the governing equation

b. In the limiting case, the form that will eliminate all sensitivity equations is dimensionless and will have eliminated (n-d) parameters from the system representation

c. The equation forms that eliminates sensitivity invariants is identical to the equation forms required by dimensional analysis to make claims about dimensional similitude

(6) Classical (dimensioned) forms of system representation are inappropriate for comparison of systems and for generalizing controller designs. Specifically:

a. Numerically equivalent differential equations (transfer functions, state-space, etc.) representations do not imply equivalent system sensitivity, and hence controller design.

b. In the reverse argument, systems that are numerically different may indeed be equivalent with respect to system sensitivity, and hence controller design.

c. The notion of dimensionless representations corrects the above flaws associated with sensitivity measures and once again allows for numerical comparisons to be made between systems.

(7) The notion of system equivalence in a dimensionless framework allows for direct and numerically measurable notions of system equivalence. Specific advantages include the following:

a. The notion of system-to-system equivalence generalizes to system-class notions of system behavior. A single 'class' of systems tend to cluster into localized regions of the dimensionless pi-space

b. Systems within a similar model class will share sensitivity invariance equations due to optimization or design constraints. These constraints generate well-defined power-law relationships between the model parameters.

c. A broad range of systems spanning mechanical and biological examples appear to exhibit mathematical properties associated with very localized dimensionless 'classes' of system representations.

d. The nominal system within a class as well as the range of behavior *over* a class is easily and numerically measurable in the dimensionless parameter domain, where it may not be obvious (or defined) in standard physical domains. The vehicle dynamics example was presented and a nominal (average) vehicle parameter was calculated from a dimensionless viewpoint.

4.9 References

1. Bode, H. W. Network Analysis and Feedback Amplifier Design. New York: Van Nostrand., 1945.
2. Bridgman, P. W. Dimensional Analysis. 3rd printing, revised ed. New Haven: Yale University Press, 1943.
3. Eslami, Mansour. Theory of Sensitivity in Dynamic Systems, An Introduction. New York: Springer-Verlag, 1994.
4. Frank, Paul M. Introduction to System Sensitivity Theory. New York: Academic Press, 1978.
5. Horowitz, I. M. Synthesis of Feedback Systems. New York: Academic Press, 1963.
6. Kline, Stephen J. Similitude and Approximation Theory. 1st ed. New York: McGraw-Hill, 1965.
7. Kokotovic, P. V., and R. S. Rutman. "Sensitivity of Automatic Control Systems." ARC 26.April (1965): 727-49.
8. Kokotovic, Petar V. Singular Perturbation Methods in Control: Analysis and Design. Orlando, Florida: Academic Press, 1986.
9. Langhaar, Henry. Dimensional Analysis and Theory of Models. 1st ed. New York: Wiley and Sons, 1951.
10. McMahon, Thomas A., and John Tyler Bonner. On Size and Life. New York: Scientific American Books, Inc., 1983.
11. Miller, K. S., and F. J. Murray. "A Mathematical Basis for an Error Analysis of Differential Analyzers." JMP 32.2 & 3 July/Oct (1953): 136-63.
12. Ngo, N. T. "Sensitivity of Automatic Control Systems." ARC 32, May.735-762 (1971).
13. Sedov, L. I. Similarity and Dimensional Methods in Mechanics. Morris Friedman. 1st ed. New York: Academic Press, 1959.

Chapter 5

Dimensional Analysis and Control: A Parametric Approach

There are several fundamental reasons for using a representation of system equations in dimensionless-form: (1) this form reduces the number of system variables as discussed in Chapter 3, (2) this form eliminates sensitivity invariants as discussed in Chapter 4, and (3) this form allows comparisons between systems of the same class as discussed in Chapter 4. Each of these advantages can be utilized in a controls context to yield results not otherwise obtainable.

The goal of this chapter is to discuss methods to simplify system/controller analysis and design using parametric methods of dimensional analysis. The central problem of control theory is to generally analyze changes in system due to variations in selected parameters (i.e. gains). Therefore, it is not surprising that dimensional analysis of parametric variations readily extends to many general control topics.

The first section of the chapter deals with the most basic problem of the stability of a characteristic equation under simultaneous parameter variations, using the vehicle control problem of Chapter 2 as an example. Next, consideration is given to specifically to the area of model reduction and simplification, which are related to parametric perturbations by the notion of λ-perturbations within the Miller-Murray classification of Chapter 4. An example system implementation is presented using a heat-exchanger system. This chapter concludes with a parametric approach to robust control. Specifically, a robust controller implementation on the vehicle control problem of Chapter 2 is presented using a parametric LMI controller synthesis.

5.1 Stability Analysis via Dimensionless Parameters

In a control theoretic context, the most basic use of a set of governing equations of a system is to analyze the stability of the system, and to determine possible transitions from stability to instability as various model parameters are changed. Indeed, the most basic purpose of a control system is to manipulate the control parameters in a manner that ensures stability of the governing equations. For a physical system where multiple parameters may be varying, incorporating dimensional constraints between the parameters can greatly simplify the analysis. As an example of this statement, the analysis of an open-loop system equation for vehicle dynamics using well-known Routh-Hurwitz criteria (Franklin, Powell, and Emami-Naeini, 2002) is presented.

The open-loop equations governing the planar dynamics of highway vehicles were presented in the second chapter. Using the Routh Stability Criterion, the classical yaw-rate stability limits of the open-loop system are investigated. The characteristic equation of the open-loop system, minus the two free integrators, is given by:

$$s^2 + \left(\frac{C_{\alpha f} + C_{\alpha r}}{mU} + \frac{a^2 \cdot C_{\alpha f} + b^2 \cdot C_{\alpha r}}{I_z \cdot U} \right) \cdot s + \frac{L^2 \cdot C_{\alpha f} \cdot C_{\alpha r}}{m \cdot I_z \cdot U^2} - \frac{a \cdot C_{\alpha f} - b \cdot C_{\alpha r}}{I_z} = 0 \qquad (5.1)$$

The system is open-loop unstable due to the free integrators (again, not shown), but in general the driver maintains closed-loop control by using steer inputs. The two free integrators correspond to the integration of lateral position to lateral velocity and the integration of yaw orientation of the vehicle to yaw rate. Aside from the two free integrators, the two remaining poles given by the characteristic equation of 5.1 are usually stable. In fact, this reduced-order characteristic equation physically represents the characteristic equation of both the yaw-rate and the lateral-acceleration dynamics.

A transition of the characteristic equation of 5.1 to instability represents a driving situation that is usually beyond the control ability of most drivers as it implies that the vehicle is unstable in the spin dynamics. Hence, an open-loop analysis of conditions introducing this type of instability is critical. To analyze the stability of the second-order characteristic equation, the Routh criterion guarantees stability if the coefficients of the characteristic equation are all greater than zero (note, Routh criteria for higher order systems are more strict than this simple positivity statement) (Franklin, Powell, and Emami-Naeini, 2002). The dimensionless form of the equation is obtained by substitution of the pi-values calculated in Chapter 2. Note that the value $\bar{s}$ is used to denote a dimensionless version of the normal Laplace variable s, where $\bar{s} = s \cdot \frac{L}{U}$:

$$\bar{s}^2 + \left(\pi_3 + \pi_4 + \frac{\pi_1^{\ 2} \cdot \pi_3 + \pi_2^{\ 2} \cdot \pi_5}{\pi_5} \right) \bar{s} + \frac{\pi_3 \cdot \pi_4 - \pi_1 \cdot \pi_3 + \pi_2 \cdot \pi_4}{\pi_5} = 0 \qquad (5.2)$$

Noting that the pi values are always positive for vehicles due to physical constraints, the only possible negative coefficient would be the last term of the polynomial. Stability limits can therefore be found by setting the last term in the denominator equal to zero. The following constraint then guarantees yaw-rate vehicle stability for the linear bicycle model.

$$\pi_3 \cdot \pi_4 - \pi_1 \cdot \pi_3 + \pi_2 \cdot \pi_4 > 0 \qquad (5.3)$$

Back substitution of the pi values yields:

$$\frac{C_{\alpha f} \cdot L}{m \cdot U^2} \cdot \frac{C_{\alpha r} \cdot L}{m \cdot U^2} - \frac{a}{L} \cdot \frac{C_{\alpha f} \cdot L}{m \cdot U^2} + \frac{b}{L} \cdot \frac{C_{\alpha r} \cdot L}{m \cdot U^2} > 0 \qquad (5.4)$$

or

$$\frac{C_{\alpha f} \cdot C_{\alpha r} \cdot L^2}{m \cdot \left(a \cdot C_{\alpha f} - b \cdot C_{\alpha r} \right)} > U_{crit}^2 \qquad (5.5)$$

This equation reproduces the classical expression for the well-known critical velocity above which an oversteer vehicle becomes unstable, and we see that the dimensionless approach already yields results in agreement with traditional approaches. Examination of the pi inequality in Equation 5.4 reveals that, from a design standpoint, the easiest way to improve stability is to increase π_4 (note that π_2 cannot be changed independent of π_1). This parameter represents the relative magnitude of the rear cornering stiffness of the vehicle. The inequality therefore suggests that rear tire adhesion should not be compromised, a well-known fact for preventing vehicle oversteer instability (Wong, 1993).

In addition to traditional results, much more can be gained from dimensional analysis. The stability criterion can be simplified by considering the information inherent in the parameter distributions discussed in Chapter 4. Knowing that $\pi_2 = 1 - \pi_1$ due to physical constraints, a relationship is sought between π_3 and π_4 (the non-dimensional cornering stiffness parameters) that simplifies the stability constraint of Equation 5.5; π_3. is plotted versus π_4 in Figure 5.1 below:

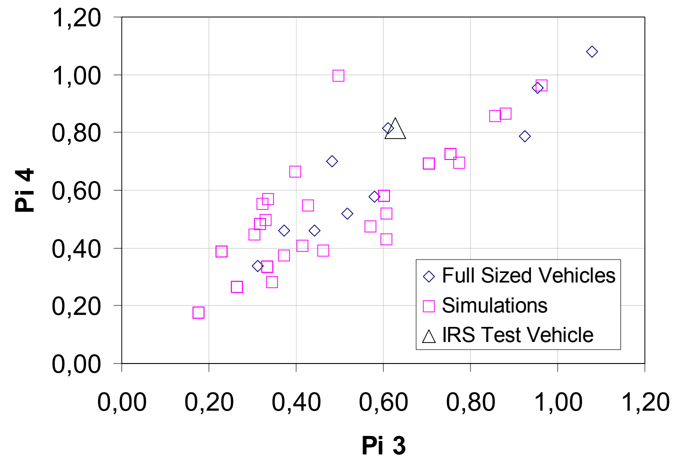


Figure 5.1: Pi4 versus Pi3, showing interrelationship

A general relationship was found to exist where $\pi_3 \cong \pi_4$ to a first approximation. Noting the physical meaning of these parameters, the cornering stiffness represents the ratio between tire force and tire sliding angle, and is related to the frictional force of the tire. The plot above indicates that the rear cornering stiffness is about the same as the front cornering stiffness, an intuitive result because both tires are made from the same material and are driving on the same road surface. This result is approximate because front and rear tire characteristics *can* vary on a vehicle as seen in Figure 5.1, so hereafter we must note that whatever stability trends we derive from the analysis are general trends, not exact numerical limits.

After assuming that the front and rear cornering stiffness are equal as a first-order approximation, the constraint on vehicle stability 5.3 can now be reduced to the expression:

$$\frac{\pi_3 + 1}{2} > \pi_1 \qquad (5.6)$$

This is simply a line dividing the π_3 versus π_4 space. A plot of experimental values of π_1 versus π_3 is shown in the figure below with the above stability line included.

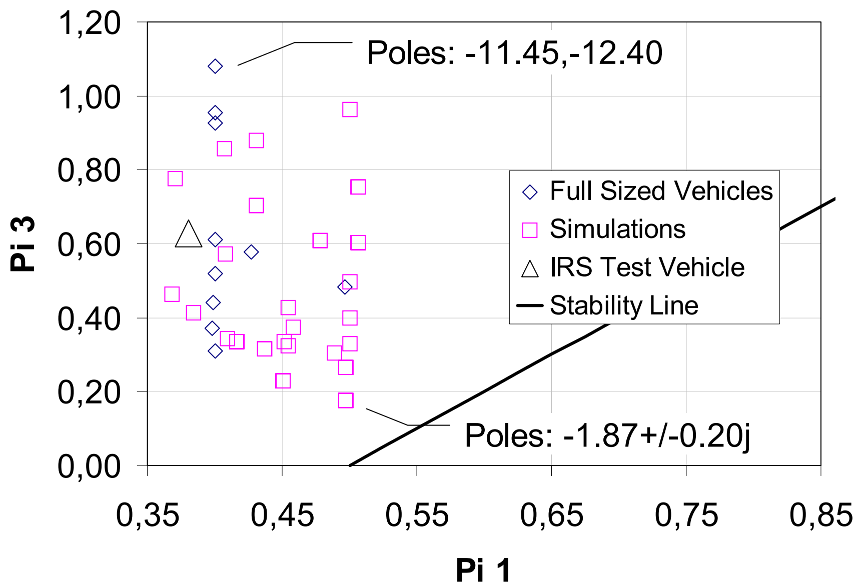


Figure 5.2: Pi1 versus Pi3 with stability line

Rather than providing a true/false statement of stability/instability, the stability line is more useful by observing parametric trends in regions near this line. From the inequality in Equation 5.6, it might be inferred that the vehicles farthest away from the stability line would likely be the most stable. To test this idea, the pi locations from the vehicles discussed in Chapter 3 are shown in the Figure 5.2. By definition, the most stable vehicles have poles farthest from the jw-axis, and the least stable vehicles have poles closest to the jw-axis. It was found that the most stable vehicle was farthest away from the above line, and the least stable vehicle was closest to the stability line.

Whereas a dimensioned approach to study Routh stability of vehicles yielded a complex, nonlinear parameter interrelationship, the dimensionless pi analysis revealed a very general, linear, almost gradient-like design criterion. It is far easier to analyze such a simple relationship versus a nonlinear one to optimize the construction of a vehicle in the sense of maximizing open-loop stability. This vehicle stability example serves to demonstrate that accounting for parameter inter-relationships, even with general approximations of interrelationship, can greatly simplify stability analysis. Moreover, this can quickly and easily be done over a range of different vehicle sizes and vehicle parameters without a significant amount of system identification. The same concept is later considered for specific use in controller design in the last section of this chapter.

5.2 Model Reduction Using Dimensionless Parameters

> Modeling for control is parsimonious and implicit. It is parsimonious, because the model should not be more detailed than that required by the specific control task. It is implicit, because the extent of the necessary detail is not known before the control task is accomplished.
>
> - Kokotovic

Chapter 4 of this thesis focused on system sensitivity from the framework of α-variations in the Miller-Murray classification of parameter variations. While this focus led to insight into sensitivity invariance, it did not address model sensitivity with regard to λ-variations, also called singular perturbations for reasons discussed shortly. These λ-variations or singular perturbations are important to the investigation of the sensitivity of a mathematical model with respect to neglected parameters or dynamics. It is common practice to reduce the model as much as possible with regard to parametric dependence and model order in order to simplify the mathematical analysis or implementation. However, there is a risk with model reduction, because the neglect of certain parameters or dynamics can give rise to considerable errors and perhaps instability if the results based on the reduced model are applied to the actual physical system (Khalil, 1996; Kokotovic, 1986; Naidu, 1988).

These aspects of parameter sensitivity are especially evident in the problem of model reduction. This section focuses on model reduction and how dimensional scaling issues arise in the analysis and methods of separating fast and slow dynamics. A more theoretic approach is presented in this section, as it found that dimensional misuse of system representations is especially common in this area. Many very knowledgeable authors are intuitively aware of this problem, and have themselves been forced to their own heuristic methods to find a dimension-independent approach. Such heuristics will be mentioned throughout this section, and a more formal justification for dimensionless techniques will be provided that should eliminate or alleviate such heuristics in the future.

It should be mentioned that the analysis presented in this section was originally motivated by problems encountered during the study of heating and cooling systems by a colleague, Bryan Rasmussen who is also a member of the Alleyne Research Group. This partnership has proved very fruitful for the author in terms of focusing the dimensional analysis study in a wider context than was originally considered, and his heating and cooling problem has generated numerous discussions on the nature of system representations and dimensional analysis in general. A detailed study of the heating/cooling problem can be found in Bryan Rasmussen's thesis completed at the University of Illinois at Urbana-Champaign in 2002 (Rassmussen, 2002). Many of the following subsections and graphs regarding the heating and cooling dynamics application are reprinted directly in his thesis with his expressed permission.

5.2.1 D.C. Motor Example

Model reduction for control design is a vast field of study. Many, if not most, of the methods currently available require the model be evaluated numerically so that appropriate state transformations or matrix operations can be used. In contrast, the singular perturbation method allows the symbolic reduction of models based on engineering knowledge of the model parameters. Singularly perturbed systems are observed in many physical systems [Naidu], including fluid dynamics, electrical circuits, aerospace systems, chemical systems, biological systems, and many others. These physical systems often contain small "parasitic" parameters that increase the dynamic order of the model. For control-oriented modeling, these parameters are generally neglected. The singular perturbation approach provides a method for justifying such assumptions, and means for analyzing the implications of these assumptions on the resulting reduced-order model. Note that references to "eliminating" dynamics means assuming that fast dynamics can be assumed to be instantaneous and thus replaced with algebraic relationships. These dynamics are not neglected, but simply replaced. The most common example system to demonstrate singular perturbation is that of the D.C. motor, and the example is given prior to the theoretical development to illustrate the intent of the analysis.

Nearly every textbook discussing singular perturbation model reduction includes a D.C. motor example (Khalil, 1996; Kokotovic, 1986). As described in Kokotovic (Kokotovic, 1986), the model consists of an equation for mechanical torque (Equation 5.1), and an equation for the electrical transient (Equation (5.3)), where i, u, R, and L are the armature current, voltage, resistance and inductance respectively, J is the moment of inertia, ω is the angular speed, and ki and $k\omega$ are the torque and back e.m.f. developed with constant excitation flux ϕ. Kokotovic asserts that in all well-designed motors the value of L is small and can be considered to be the perturbation parameter.

$$J\dot{\omega} = ki \tag{5.7}$$

$$L\dot{i} = -k\omega - Ri + u \tag{5.8}$$

Assuming that L is zero, Equation (5.3) reduces to an algebraic constraint (Equation (5.6)):

$$i = \frac{u - k\omega}{R} \tag{5.9}$$

after substitution into Equation (5.3), the resulting equation becomes the well-known first order model of the DC motor (Equation 5.4).

$$J\dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u \tag{5.10}$$

Khalil suggests that it is preferable to choose the perturbation parameter as dimensionless combination of physical parameters (Khalil, 1996). He approaches the above example by first defining several dimensionless variables as $\omega_r = \frac{\omega}{\Omega}$, $i_r = \frac{iR}{k\Omega}$, and $u_r = \frac{u}{k\Omega}$, and then rewriting Equations 5.1 and (5.3) as Equations 5.5 and 5.3, where $T_e = L/R$ is the electrical time constant, and $T_m = JR/k^2$ is the mechanical time constant.

$$T_m \frac{d\omega_r}{dt} = i_r \tag{5.11}$$

$$T_e \frac{di_r}{dt} = -\omega_r - i_r + u_r \tag{5.12}$$

Assuming that $T_e << T_m$ and defining the dimensionless time variable $t_r = t/T_m$, the state equations can be rewritten as Equations 5.6 and (5.4). The ratio T_e/T_m then becomes the obvious choice for the perturbation parameter.

$$\frac{d\omega_r}{dt_r} = i_r \tag{5.13}$$

$$\left(\frac{T_e}{T_m}\right)\frac{di_r}{dt_r} = -\omega_r - i_r + u_r \tag{5.14}$$

For the benefit of future discussion, Equations 5.6 and (5.4) are written in state space format (Equation 5.1). The eigenvalues of the full-order system can be computed symbolically (Equation .2).

$$\begin{bmatrix} \dot{\omega}_r \\ \dot{i}_r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{T_m} \\ -\frac{1}{T_e} & -\frac{1}{T_e} \end{bmatrix}}_{A} \begin{bmatrix} \omega_r \\ i_r \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{T_e} \end{bmatrix}}_{B} u_r \qquad (5.15)$$

$$\begin{aligned} \lambda(A) &= \frac{1}{2}\left[-\frac{1}{T_e} \pm \sqrt{\frac{1}{T_e^{\,2}} - \frac{4}{T_e T_m}} \right] \\ &= \frac{1}{2T_e}\left[-1 \pm \sqrt{1-4\varepsilon} \right] \end{aligned} \qquad (5.16)$$

The ratio of the eigenvalues is easily computed as well (Equation 5.2). Assuming $\sqrt{1-4\varepsilon} \approx 1$, the perturbation parameter is found to be the ratio of the eigenvalues (Equation 5.18). This is in agreement with Kokotovic who notes that the perturbation parameter is on the order of the ratio of the slow and fast eigenvalues. This also demonstrates that choosing ε as a dimensionless parameter is *preferable* because the ratio of eigenvalues is always dimensionless.

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &= \frac{1-\sqrt{1-4\varepsilon}}{1+\sqrt{1-4\varepsilon}} \\ &= \frac{4\varepsilon}{\left(1+\sqrt{1-4\varepsilon}\right)^2} \end{aligned} \qquad (5.17)$$

$$\frac{\lambda_1}{\lambda_2} \approx \varepsilon \qquad (5.18)$$

Note that Khalil's method implicitly involves dimensional analysis, since the use of a dimensionless time and dimensionless states amounts to forming a dimensionless dynamic system. The justification for this approach is fairly weak, but it does imply that Khalil's experience with the dimensional form of singular perturbation encountered problems that were best solved by a dimensionless approach. Khalil's method of dimensional normalization is common (for instance, see Chapter 1 of Skogestad and Postlethwaite (Skogestad and Postlethwaite, 2000)), but is clearly suboptimal because there may be no reduction in parameters if the states are each normalized independently. A better method based directly from a dimensional analysis is suggested in a later example.

5.2.2 Dimensional Constraints on Analytic Methods of Model Reduction

In a parameter-sensitivity context, a singularly perturbed continuous system is assumed to be perturbed by only one single λ-variation. The perturbation parameter, ε is chosen to represent the λ-variation in the equation, and system description is given by the general form of a vector differential equation by 5.19 .(Kokotovic, 1986; Frank, 1978; Sastry, 1999)

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x},\mathbf{z},t,\mathbf{u},\varepsilon\right), \mathbf{x}\left(t_0\right) = x_0 \\ \varepsilon \cdot \dot{\mathbf{z}} &= \mathbf{f}_1\left(\mathbf{x},\mathbf{z},t,\mathbf{u},\varepsilon\right), \mathbf{z}\left(t_0\right) = z_0 \\ \mathbf{y} &= \mathbf{g}\left(\mathbf{x},\mathbf{z},t,\mathbf{u},\varepsilon\right) \end{aligned} \tag{5.19}$$

The nominal value of ε is assumed to be $\varepsilon_0 = 0$, and it is clear that the order of the equation changes at the nominal value of ε. The term $\mathbf{x} = \left[x_1 \quad x_2 \quad \ldots \quad x_n\right]^T$ is a $n \times 1$ state vector and $\mathbf{z} = \left[x_{n+1} \quad x_{n+2} \quad \ldots \quad x_{n+r}\right]^T$ is a $r \times 1$ state vector which represents the increase in system order when $\varepsilon \neq 0$. The output equation $\mathbf{y}$ is a $q \times 1$ output vector. The functions $\mathbf{f}, \mathbf{f}_1, \mathbf{g}$ are n-, r-, or q-dimensional vector functions respectively. Equation 5.19 will be called the actual state and output equations. We now define the degenerate equation by the examining the solution to 5.19 under the assumption $\varepsilon = 0$. The number of states is diminished from:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \left[x_1 \quad x_2 \quad \ldots \quad x_{n+r}\right]^T \tag{5.20}$$

to

$$\mathbf{x}_0 = \left[x_{10} \quad x_{20} \quad \ldots \quad x_{n0}\right]^T \tag{5.21}$$

The state equations become:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}_0, \mathbf{z}_0, t, \mathbf{u}, 0\right) \tag{5.22}$$

$$0 = \mathbf{f}_1\left(\mathbf{x}_0, \mathbf{z}_0, t, \mathbf{u}, 0\right) \tag{5.23}$$

At least one real solution to Equation 5.24 is assumed to exist and is given by:

$$\mathbf{z}_0 = \varphi\left(\mathbf{x}_0, t, \mathbf{u}\right) \tag{5.25}$$

Note that the initial value of this equation may be different from the original, actual-state equation. If the solution is substituted into Equation 5.1, the degenerate model becomes:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}_0, \varphi\left(\mathbf{x}_0, t, \mathbf{u}\right), t, \mathbf{u}, 0\right) \\ &= \mathbf{f}_0\left(\mathbf{x}_0, \mathbf{u}, t\right) \end{aligned} \tag{5.26}$$

The state $\mathbf{z}$ disappears completely from the degenerate model, and the order of the model is observed to be reduced by r. In the transition as $\varepsilon \to 0$, we find that the differential equation:

$$\dot{\mathbf{z}} = \frac{1}{\varepsilon}\mathbf{f}_1\left(\mathbf{x}, \mathbf{z}, t, \mathbf{u}, \varepsilon\right) \tag{5.27}$$

tends to infinity. It is for this reason that the equation is called a singularly perturbed system.

Some have remarked that the value of ε is approximately the ratio of the slow eigenvalues to the fast eigenvalues (Naidu, 1988). While this is certainly true for the motor case, this statement is meaningless if there is more than one eigenvalue associated with either the $\mathbf{x}$ or $\mathbf{z}$ subspaces. Naidu notes that ε represents an intrinsic property of the system and does not necessarily have to appear explicitly in the system. For many systems, an explicit choice of the perturbation parameter may not be possible for complex physical models where the perturbation parameter may be implicit, or the fast phenomenon unknown. Clearly however, a only a dimensionless ratio can serve as the perturbation parameter; otherwise, one might make it arbitrarily small or large by simple changes in the units of the problem, and any attempted justification that the parameter is 'large' or 'small' would be irrelevant.

The techniques for applying the singular-perturbation model reduction method discussed in this thesis will involve linear time invariant models of singularly perturbed systems. The states $\mathbf{x}$ are assumed to represent the slow dynamics and the states $\mathbf{z}$ to represent the fast dynamics of the system. Under the assumption of linearity, the system model of Equation 5.19 becomes:

$$
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}
$$

$$
\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{D} \end{bmatrix} \mathbf{u} \tag{5.28}
$$

Dimensionally, this state-space form requires:

$$
\begin{aligned}
\mathbf{A}_{11} &\mp \mathbf{x} \cdot \mathbf{x}^{[-1]} / t_1 \\
\mathbf{A}_{12} &\mp \mathbf{x} \cdot \mathbf{z}^{[-1]} / t_1 \\
\mathbf{A}_{21} &\mp \mathbf{z} \cdot \mathbf{x}^{[-1]} / t_2 \\
\mathbf{A}_{22} &\mp \mathbf{z} \cdot \mathbf{z}^{[-1]} / t_2 \\
\mathbf{B}_1 &\mp \mathbf{x} \cdot \mathbf{u}^{[-1]} / t_1 \\
\mathbf{B}_2 &\mp \mathbf{z} \cdot \mathbf{u}^{[-1]} / t_2 \\
\mathbf{C}_1 &\mp \mathbf{y} \cdot \mathbf{x}^{[-1]} \\
\mathbf{C}_2 &\mp \mathbf{y} \cdot \mathbf{z}^{[-1]} \\
\mathbf{D} &\mp \mathbf{y} \cdot \mathbf{u}^{[-1]}
\end{aligned} \tag{5.29}
$$

Numerically, we wish to separate the fast and slow dynamics, but any numerical operations on the system representation will necessarily require that the above dimensional constraints will be met. Therefore, we consider numerical techniques of separating the system into fast and slow dynamics. There are two primary methods that are in common use to numerically and/or symbolically simplify the singularly perturbed system: (1) using a residualization method, or (2) using a balancing transformation to create a diagonal form.

Residualization

The residualization method, as referred to by Skogestad and Postlethwaite (Skogestad and Postlethwaite, 2000), is to approximate the dynamics by the limiting case as the perturbation parameter becomes infinite. This corresponds to simply setting $\dot{z} = 0$ in Equation 5.28. The resulting algebraic equation can be solved for $\mathbf{z}$ in terms of $\mathbf{x}$ and substituted in to the remaining differential equation. The resulting formulas for the reduced order state space model are given below in Equation 5.30.

$$\begin{aligned} \mathbf{A}_r &= \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \\ \mathbf{B}_r &= \mathbf{B}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_2 \\ \mathbf{C}_r &= \mathbf{C}_1 - \mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \\ \mathbf{D}_r &= \mathbf{D} - \mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2 \end{aligned} \qquad (5.30)$$

We now consider the dimensional requirements of the above transformation. The equation $\mathbf{A}_r = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ dimensionally requires that:

$$\begin{aligned} \mathbf{A}_r &\mp \mathbf{A}_{11} \\ &\mp \mathbf{x}\cdot\mathbf{x}^{[-1]}/t_1 \\ &\mp \underbrace{\mathbf{x}\cdot\mathbf{z}^{[-1]}/t_1}_{\mathbf{A}_{12}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{z}^{[-1]}\cdot t_2}_{\mathbf{A}_{22}^{-1}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{x}^{[-1]}/t_2}_{\mathbf{A}_{21}} \end{aligned} \qquad (5.31)$$

The equation $\mathbf{B}_r = \mathbf{B}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_2$ dimensionally requires that:

$$\begin{aligned} \mathbf{B}_r &\mp \mathbf{B}_1 \\ &\mp \mathbf{x}\cdot\mathbf{u}^{[-1]}/t_1 \\ &\mp \underbrace{\mathbf{x}\cdot\mathbf{z}^{[-1]}/t_1}_{\mathbf{A}_{12}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{z}^{[-1]}\cdot t_2}_{\mathbf{A}_{22}^{-1}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{u}^{[-1]}/t_2}_{\mathbf{B}_2} \end{aligned} \qquad (5.32)$$

The equation $\mathbf{C}_r = \mathbf{C}_1 - \mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ dimensionally requires that:

$$\begin{aligned} \mathbf{C}_r &\mp \mathbf{C}_1 \\ &\mp \mathbf{y}\cdot\mathbf{x}^{[-1]} \\ &\mp \underbrace{\mathbf{y}\cdot\mathbf{z}^{[-1]}}_{\mathbf{C}_2}\cdot\underbrace{\mathbf{z}\cdot\mathbf{z}^{[-1]}\cdot t_2}_{\mathbf{A}_{22}^{-1}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{x}^{[-1]}/t_2}_{\mathbf{A}_{21}} \end{aligned} \qquad (5.33)$$

Finally, the equation $\mathbf{D}_r = \mathbf{D} - \mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2$ dimensionally requires that:

$$\begin{aligned} \mathbf{D}_r &\mp \mathbf{D} \\ &\mp \mathbf{y}\cdot\mathbf{u}^{[-1]} \\ &\mp \underbrace{\mathbf{y}\cdot\mathbf{z}^{[-1]}}_{\mathbf{C}_2}\cdot\underbrace{\mathbf{z}\cdot\mathbf{z}^{[-1]}\cdot t_2}_{\mathbf{A}_{22}^{-1}}\cdot\underbrace{\mathbf{z}\cdot\mathbf{u}^{[-1]}/t_2}_{\mathbf{B}_2} \end{aligned} \qquad (5.34)$$

Each of the above transformation equations is by inspection dimensionally consistent if the original equation is dimensionally consistent. It should be clear that the method of residualization is therefore defined dimensionally as long as the perturbation parameter is dimensionless.

Diagonalization

Another method of simplifying a singularly perturbed system is to mathematically decouple the fast and slow dynamics, such that the system can be represented as Equation (5.35).

$$\begin{bmatrix} \dot{\mathbf{x}}_s \\ \dot{\mathbf{z}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & 0 \\ 0 & \mathbf{A}_f \end{bmatrix} \begin{bmatrix} \mathbf{x}_s \\ \mathbf{z}_f \end{bmatrix} + \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B}_f \end{bmatrix} \cdot \mathbf{u} \tag{5.35}$$

The above state-space representation implicitly necessitates the following dimensional constraints:

$$\begin{aligned} \mathbf{A}_s &\mathrel{\overline{\overline{\vee}}} \mathbf{x}_s \cdot \mathbf{x}_s^{[-1]} / t_1 \\ \mathbf{A}_f &\mathrel{\overline{\overline{\vee}}} \mathbf{z}_f \cdot \mathbf{z}_f^{[-1]} / t_2 \\ \mathbf{B}_s &\mathrel{\overline{\overline{\vee}}} \mathbf{x}_s \cdot \mathbf{u}^{[-1]} / t_1 \\ \mathbf{B}_f &\mathrel{\overline{\overline{\vee}}} \mathbf{z}_f \cdot \mathbf{u}^{[-1]} / t_2 \end{aligned} \tag{5.36}$$

The diagonalization (5.35) of the original system represented in Equations (5.28) is achieved by applying the transformations of Equation (5.17). Interestingly, no matrix inversion is necessary for calculating $\mathbf{T}^{-1}$ in Equation 5.37. The matrices $\mathbf{L}$ and $\mathbf{M}$ are found as the solution to the Ricatti Equations (5.41) and (5.42). Further explanation of this technique, as well as a proof of its validity can be found in (Kokotovic, 1986; Naidu, 1988; Sastry, 1999).

$$\begin{aligned} \mathbf{A}_s &= \mathbf{A}_{11} - \mathbf{A}_{12} \cdot \mathbf{L} \\ \mathbf{A}_f &= \mathbf{A}_{22} + \mathbf{L} \cdot \mathbf{A}_{12} \\ \mathbf{B}_s &= \mathbf{B}_1 - \mathbf{M} \cdot \mathbf{L} \cdot \mathbf{B}_1 - \mathbf{M} \cdot \mathbf{B}_2 \\ \mathbf{B}_f &= \mathbf{B}_2 - \mathbf{L} \cdot \mathbf{B}_1 \end{aligned} \tag{5.38}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_1 - \mathbf{M} \cdot \mathbf{L} & -\mathbf{M} \\ \mathbf{L} & \mathbf{I}_2 \end{bmatrix} \tag{5.39}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{M} \\ -\mathbf{L} & \mathbf{I}_2 - \mathbf{L} \cdot \mathbf{M} \end{bmatrix} \tag{5.40}$$

$$\mathbf{L}\cdot\mathbf{A}_{11}+\mathbf{A}_{21}-\mathbf{L}\cdot\mathbf{A}_{12}\cdot\mathbf{L}-\mathbf{A}_{22}\cdot\mathbf{L}=0 \qquad (5.41)$$

$$\mathbf{A}_s\cdot\mathbf{M}-\mathbf{M}\cdot\mathbf{A}_f+\mathbf{A}_{12}=0 \qquad (5.42)$$

Note that we do not yet know dimensional requirements on the matrices, $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{T}$. Because these matrices are multipliable, we know from Chapter 2 that we may represent them by left and right dimensional vectors denoted by subscript L and R, i.e.:

$$\begin{aligned}\mathbf{L}&\mp\mathbf{l}_L\cdot\mathbf{l}_R^{[-1]}\\ \mathbf{M}&\mp\mathbf{m}_L\cdot\mathbf{m}_R^{[-1]}\\ \mathbf{T}&\mp\mathbf{t}_L\cdot\mathbf{t}_R^{[-1]}\end{aligned} \qquad (5.43)$$

We now consider the dimensional constraints of the above equations. The equation, $\mathbf{A}_s=\mathbf{A}_{11}-\mathbf{A}_{12}\cdot\mathbf{L}$, implies the dimensional constraints:

$$\begin{aligned}\mathbf{A}_s&\mp\mathbf{A}_{11}\\ &\mp\mathbf{x}\cdot\mathbf{x}^{[-1]}/t_1\\ &\mp\underbrace{\mathbf{x}\cdot\mathbf{z}^{[-1]}/t_1}_{\mathbf{A}_{12}}\cdot\underbrace{\mathbf{l}_L\cdot\mathbf{l}_R^{[-1]}}_{L}\end{aligned} \qquad (5.44)$$

We can immediately note that, for dimensional consistency of addition, we require that:

$$\mathbf{l}_L\cdot\mathbf{l}_R^{[-1]}\mp\mathbf{z}\cdot\mathbf{x}^{[-1]} \qquad (5.45)$$

The next equation, $\mathbf{A}_f=\mathbf{A}_{22}+\mathbf{L}\cdot\mathbf{A}_{12}$, implies the dimensional constraints:

$$\begin{aligned}\mathbf{A}_f&\mp\mathbf{A}_{22}\\ &\mp\mathbf{z}\cdot\mathbf{z}^{[-1]}/t_2\\ &\mp\underbrace{\mathbf{l}_L\cdot\mathbf{l}_R^{[-1]}}_{L}\cdot\underbrace{\mathbf{x}\cdot\mathbf{z}^{[-1]}/t_1}_{\mathbf{A}_{12}}\end{aligned} \qquad (5.46)$$

We can immediately note that, for dimensional consistency of addition, we require that:

$$\mathbf{l}_L\cdot\mathbf{l}_R^{[-1]}\mp\mathbf{z}\cdot\mathbf{x}^{[-1]}t_1/t_2 \qquad (5.47)$$

Comparing (5.3) and (5.6), the two transformations are only dimensionally consistent if:

$$t_1 \mp t_2 \qquad (5.48)$$

The requirement of Equation (5.6) then becomes:

$$\mathbf{l}_L \cdot \mathbf{l}_R^{[-1]} \mp \mathbf{z} \cdot \mathbf{x}^{[-1]} \qquad (5.49)$$

The transformation $\mathbf{B}_s = \mathbf{B}_1 - \mathbf{M} \cdot \mathbf{L} \cdot \mathbf{B}_1 - \mathbf{M} \cdot \mathbf{B}_2$ requires the following for dimensional consistency:

$$\begin{aligned} \mathbf{B}_s &\mp \mathbf{B}_1 \\ &\mp \mathbf{x} \cdot \mathbf{u}^{[-1]} / t_1 \\ &\mp \underbrace{\mathbf{m}_L \cdot \mathbf{m}_R^{[-1]}}_{M} \cdot \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R^{[-1]}}_{L} \cdot \underbrace{\mathbf{x} \cdot \mathbf{u}^{[-1]} / t_1}_{B_1} \\ &\mp \underbrace{\mathbf{m}_L \cdot \mathbf{m}_R^{[-1]}}_{M} \cdot \underbrace{\mathbf{z} \cdot \mathbf{u}^{[-1]} / t_2}_{B_2} \end{aligned} \qquad (5.50)$$

Substituting relationships previously required from Equations 5.4 and 5.5, this requires the additional relation:

$$\mathbf{m}_L \cdot \mathbf{m}_R^{[-1]} \mp \mathbf{x} \cdot \mathbf{z}^{[-1]} \qquad (5.51)$$

The next relation $\mathbf{B}_f = \mathbf{B}_2 - \mathbf{L} \cdot \mathbf{B}_1$ requires the following for dimensional consistency:

$$\begin{aligned} \mathbf{B}_f &\mp \mathbf{B}_2 \\ &\mp \mathbf{z} \cdot \mathbf{u}^{[-1]} / t_2 \\ &\mp \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R^{[-1]}}_{L} \cdot \underbrace{\mathbf{x} \cdot \mathbf{u}^{[-1]} / t_1}_{B_1} \end{aligned} \qquad (5.52)$$

We find that this is satisfied under the constraints of Equation 5.4 and 5.5. The relation $\mathbf{L} \cdot \mathbf{A}_{11} + \mathbf{A}_{21} - \mathbf{L} \cdot \mathbf{A}_{12} \cdot \mathbf{L} - \mathbf{A}_{22} \cdot \mathbf{L} = 0$ dimensionally requires that:

$$
\begin{aligned}
\mathbf{A}_{21} &\mp \mathbf{z} \cdot \mathbf{x}^{[-1]} / t_2 \\
&\mp \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R{}^{[-1]}}_{L} \cdot \underbrace{\mathbf{x} \cdot \mathbf{x}^{[-1]} / t_1}_{A_{11}} \\
&\mp \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R{}^{[-1]}}_{L} \cdot \underbrace{\mathbf{x} \cdot \mathbf{x}^{[-1]} / t_1}_{A_{12}} \cdot \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R{}^{[-1]}}_{L} \\
&\mp \underbrace{\mathbf{z} \cdot \mathbf{z}^{[-1]} / t_2}_{A_{22}} \cdot \underbrace{\mathbf{l}_L \cdot \mathbf{l}_R{}^{[-1]}}_{L}
\end{aligned}
\qquad (5.53)
$$

Which is again satisfied under the constraints of Equation 5.4 and 5.5. Finally, we require that $\mathbf{A}_s \cdot \mathbf{M} - \mathbf{M} \cdot \mathbf{A}_f + \mathbf{A}_{12} = 0$ satisfy the dimensional constraints:

$$
\begin{aligned}
\mathbf{A}_{12} &\mp \mathbf{x} \cdot \mathbf{z}^{[-1]} / t_1 \\
&\mp \underbrace{\mathbf{x}_s \cdot \mathbf{x}_s{}^{[-1]} / t_1}_{A_s} \cdot \underbrace{\mathbf{m} \cdot \mathbf{m}^{[-1]}}_{M} \\
&\mp \underbrace{\mathbf{m} \cdot \mathbf{m}^{[-1]}}_{M} \cdot \underbrace{\mathbf{z}_f \cdot \mathbf{z}_f{}^{[-1]} / t_2}_{A_f}
\end{aligned}
\qquad (5.54)
$$

Which is satisfied by the constraints of Equations 5.4 and 5.51.

In summary, two reduction techniques were presented: residualization and balanced truncation. Residualization may be difficult to use in practice because it may be difficult to find a physically meaningful perturbation parameter. The diagonalization approach may not be useful as an analytic method (versus a numerical approach) because much of the meaning of the model is lost after the balancing operations. Dimensionally both are well defined under certain assumptions, as follows:

Residualization Dimensional Constraints:

(1) The perturbation parameter, ε, must be dimensionless

Diagonalization Dimensional Constraints:

(1) The time dimension must be consistent throughout the model representation, i.e. $t_1 \mp t_2$

(2) The transformation matrices must have physical dimensions: $\mathbf{l}_L \cdot \mathbf{l}_R^{[-1]} \mp \mathbf{z} \cdot \mathbf{x}^{[-1]}$, and $\mathbf{m}_L \cdot \mathbf{m}_R^{[-1]} \mp \mathbf{x} \cdot \mathbf{z}^{[-1]}$.

5.2.3 Methods of Model Comparison

In order to produce a model reduction that is analytically meaningful from a higher-order first-principles derivation, a method of residualization is obviously preferred of the method of diagonalization discussed previously. Unfortunately, there may be a multitude of parameter choices or parameter ratios to choose for the singular perturbation parameter (or λ-parameter in the Miller-Murray classification). This problem may be resolved by noting that, for most physical systems, there is generally a much smaller subset of possible choices of meaningful state-variables. Therefore, a problem of choosing among a multitude of possible perturbation parameters to determine which ones produce singular behavior is simplified to choosing between a small set of different state-representations.

The goal of model reduction is therefore reframed as the following problem: Given a set of equivalent system representations, each with different physical meaning of states and each assumed to be singularly perturbed by an unknown λ-parameter, determine which representation is most appropriate for a residualization analysis and determine which states within that representation should be eliminated. The answer to this question will obviously involve a comparison across different state-variable representations, each involving different physical unit systems. Hence, careful consideration must be given to the dimensional validity of functions used for making such comparisons.

Intuition dictates that the most ideal representation for residualization is a modal form where the fast/slow states are not coupled and off-diagonal terms are small. Thus the fast/slow dynamics are explicitly associated with particular states, and the choice of states to residualize is then obvious. Alternatively, a representation that is either upper or lower diagonal is preferable because the off-diagonal terms would not affect the eigenvalues. These off-diagonal terms would, however, affect the conditioning of the matrix and possibly the reduced order model approximation. Thus given several acceptable model representations, the "best" choice for residualization would be the representation that is diagonally dominant or block diagonally dominant with the fast states decoupled from the slow states. There are several methods available for measuring the relative coupling of the dynamics; these include diagonal dominance, induced matrix norms, and the relative gain array. Block diagonal dominance is much more difficult to measure numerically, and is not discussed. If the system is not diagonal dominant, an alternative means of finding the residualized states would be to find an appropriate scaling matrix to form a balanced realization. From examination of the scaling matrix, it might be inferred which states are associated with the degenerate model and which are residual.

The traditional mathematical definition of diagonal dominance is given in Equation 5.1. In words, a matrix is diagonally dominant in the sense that the absolute value of the diagonal element of each row is strictly greater than the sum of the absolute values of the off-diagonal elements. To be specific, this is termed row diagonal dominance. Column diagonal dominance is similarly computed, but not considered here.

$$|a_{ii}| \geq \sum_{i \neq j} |a_{ij}| \tag{5.42}$$

We may immediately note that this comparison only is relevant for a small class of dimensional representations. Specifically, we require that the dimensions of all the elements of the system matrix be equivalent, a type of physical dimensional constraint called a dimensionally uniform matrix. Uniform matrices are dimensionally parallel to dimensionless matrices. Therefore, we may state the following important point:

Diagonal dominance is only defined for dimensionless system representations and representations that are a constant gain multiplied by the dimensionless system representation.

Several of the traditional induced matrix norms also give a measure of diagonal dominance. These are defined in Equations 5.43 - 5.45 as the induced one norm (maximum column sum), the induced infinity norm (maximum row sum), and the induced two norm (maximum singular value). The minimal value of each these norms will occur for a strictly diagonal representation, with the minimal value being equal to the largest eigenvalue (Equation 5.46) (Proof in (Skogestad and Postlethwaite, 2000)). Thus these norms can provide a means of comparing the diagonal dominance of different representations. For the example presented shortly, the induced two-norm (maximum singular value) is used.

$$\|A\|_{i1} = \max_j \left(\sum_i |a_{i,j}| \right) \tag{5.43}$$

$$\|A\|_{i\infty} = \max_i \left(\sum_j |a_{i,j}| \right) \quad (5.44)$$

$$\|A\|_{i2} = \sigma(A) = \sqrt{\max(\lambda(A^T A))} \quad (5.45)$$

$$\max(\lambda(A)) \leq \|A\| \quad (5.46)$$

If we analyze the norms dimensionally, we find that the first three norms are only defined for uniform matrices (see discussion in (Hart, 1995)), while the last norm is dimensionally defined for all matrices.

Matrix norms are in general only defined for dimensionless system representations and representations that are a constant gain multiplied by a dimensionless system representation.

Singularly perturbed systems generally involve system matrices that are ill conditioned, where the condition number of a matrix is the ratio of the largest singular value to the smallest singular value. A system that exhibits multiple time scale behavior will have eigenvalues that differ by orders of magnitude, and generally a large condition number (ratio of largest to smallest singular values).

The notion of condition number requires a discussion of singular-value analysis from a dimensional viewpoint. The dimensional requirements for singular-value decomposition are rather strict and described in detail by Hart (Hart, 1995). A singular-value decomposition of a matrix is given as:

$$\mathbf{A} = \mathbf{U} \cdot \mathbf{D} \cdot \mathbf{V}^T \quad (5.47)$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal and consist of orthonormal vectors. However, one requirement for the matrices to be orthogonal and orthonormal is that the matrices are dimensionless. A lengthy discussion is provided in Hart regarding this topic. The argument can be understood by noting that the singular-value decomposition is related to the spectral norm given by Rayleigh's principle. Rayleigh's principle states that the expression:

$$\left|\frac{\mathbf{A}\cdot\mathbf{x}}{\mathbf{x}}\right| \tag{5.48}$$

is maximized when $\mathbf{x}$ is in the direction of the first column of the matrix $\mathbf{V}$ from Equation 5.47. The ratio then takes on the value of σ_1, which is the spectral norm of $\mathbf{A}$. This corresponds to the maximum possible amplification possible by $\mathbf{A}$, where amplification is measured as a ratio of magnitudes in the domain and range space. However, magnitudes are only defined in dimensionless spaces. We could modify this by using a norm-space rather than magnitude, which relaxes the dimensional requirements on $\mathbf{A}$ to be uniform. Therefore, we see that the matrix A must be dimensionless or uniform for the singular-value decomposition to apply to A. Both conditions are highly restrictive and are usually violated in the standard usage of SVD.

5.2.3.1 Scaling Matrix

For systems that are not diagonally dominant, but still exhibit multiple time scale behavior, an alternative method for selecting the states to be residualized is needed. One technique is to form the scaling matrix necessary to form a balanced realization, then examine this matrix to observe which rows or columns are being weighted most heavily. In the context of this thesis, balancing the system matrices is performed by finding a scaling matrix S such that the norm of Equation 5.49 is minimized. When calculating the scaling matrix, the entries of S are generally restricted to integer exponents of 2 so that computation errors are not introduced. By evaluating the diagonal entries of S, appropriate choices of which states should be residualized can be made. This is equivalent to visually inspecting the matrix and determining that the entries of a specific row are an order of magnitude higher than the other rows. This row is assumed to be multiplied by $1/\varepsilon$. Thus by dividing this row by ε, the system of equations are then made into the standard form of a singularly perturbed system discussed earlier. The elements of the scaling matrix give a numerical measure for which rows have entries that are relatively large.

For this method the best representation for model reduction could be chosen as the representation with the lowest condition number (the least ill-conditioned).

$$\begin{bmatrix} S^{-1}AS & S^{-1}B \\ CS & 0 \end{bmatrix} \tag{5.49}$$

After determining the best choice of the available representations for model reduction, the states are reordered into the standard form presented earlier. The number of desired states is residualized according to the technique presented previously, and the approximated eigenvalues can be compared to the full order eigenvalues.

In nearly all of the methods of model comparison, the dimensional requirements for model comparison are rather strict. Specifically, we found that both the residualization process and the diagonalization process were dimensionally constrained to uniform time units within the physical representation. We also found find a similar but stronger constraint for the model comparisons, that the system matrices must be uniform or dimensionless in order for many of the common comparison mathematics to be valid.

The subset of matrices that are dimensionally valid arguments to many of the above measures is an exceedingly small set of physical representations. In general, dimensional constraints are violated without the user's knowledge of the error involved. It is interesting some control designers that are familiar with system operations have developed empirical techniques to address this issue. For instance, in the first Chapter of the book by Skogestad and Postlethwaite and in the singular perturbation chapters of Khalil (Khalil, 1996; Skogestad and Postlethwaite, 2000), empirical approaches are recommended that implicitly form dimensionless or uniform system representations. These empirical methods do not utilize the pi-approach to form the models, but rather use division by state-norms or arbitrary parameters to achieve a renormalization of the problem. While the empirical methods certainly correct the dimensional problems that would otherwise be encountered in application of the above mathematics, it does not address the possible sensitivity invariants discussed in Chapter 4, or the parameter reduction discussed in Chapter 2.

We therefore develop a methodology for model reduction based on a dimensionless system representation based on Dimensional Analysis. The steps would be as follows:

(1) Using one of the several methods presented in this thesis, generate a dimensionless model representation from the original model
(2) Use the necessary tools described above to analyze this model and perform the needed model reduction
(3) Observe what states are being eliminated in the dimensionless model reduction, and use the dimensionless model form as a template to eliminate states in the standard, dimensioned form of the model.

After the chosen representation has been residualized, the eigenvalues of the reduced order model can be compared to those of the full order model. The approximation error can be calculated and verified to be within acceptable limits. Additionally, many physical insights can be gained by evaluating which physical states are associated with fast dynamics, and which are associated with slow dynamics. This method is demonstrated in the following section using a complex heating and cooling system at the University of Illinois as an example.

5.3 Model Reduction Example: Heating and Cooling System

The previous section presented a multitude of numerical model reduction techniques that can be performed with linearized models. The approaches to be considered in this example are restricted to those that preserve the physical meaning of the dynamic states so that the reduced-order model can be used to determine methods to improve the design and control of the physical system.

To describe the problem setup leading up to the model reduction problem, an analytical first-principles model of subcritical and transcritical components in several different air-conditioning systems has been derived (Rassmussen, 2002). These nonlinear component-based models were validated with experimental data. These component models result in a system model that is high order and nonlinear, so linearized models of both the system and individual components were obtained analytically. A numerical analysis via canonical methods (balancing, controllable canonic forms, modal analysis, etc.) reveals that the system exhibits a significant difference in time-scale between fast and slow dynamics. It also revealed a redundant dynamic constraint that produces an artificial integrator in the system description. Empirical models constructed by numerical analysis demonstrated that lower order models were sufficient for predicting the dominant dynamic behavior of the system. Furthermore, analysis of the linearized version of the derived models also indicated that the dominant dynamic behavior could be captured with a low order model. However, analytic meaning of the reduced-order model behavior was lost. Due to the complexity and interaction of the model components, it was unclear which physical components correspond to either the fast or slow dynamics. To understand the problem further, three different state representations were obtained where each representation has well-known physical meaning for the states. A reduced-order, physics-based model is desired for one of the three representations.

The model reduction problem is then given as follows: (1) Given several choices of state-representations, which will yield the best reduced order model approximation of the full order system? (2) Given the chosen representation, which states should be considered fast/slow? (3) Does the resulting reduced order model adequately approximate the full order model? And (4), what is the physical interpretation of the choices of the fast/slow states?

A method to solve this problem was outlined in the earlier section. First, each of the possible representations for the component models is put into dimensionless form for reasons of model comparison described earlier. These dimensionless component-wise representations are evaluated to determine the most suitable representation for model reduction, as well as which states should be residualized. Reduced-order component models are calculated and compared to the full order models. Observations regarding the physical meaning of the negligible dynamics are made. The reduced-order models are then combined to create a reduced-order system model. This model is then compared to the full-order nonlinear and linearized analytical system models through analysis and simulation. Comparisons with the experimental system are also made. Each component-based analysis is discussed below:

5.3.1 Gas Cooler Order Reduction

For the gas cooler, there are three possible choices of states for the gas cooler. They are given as:

(1) $$x = \begin{bmatrix} P_c & h_c & T_w \end{bmatrix}^T \qquad (5.50)$$

(2) $$x' = \begin{bmatrix} P_c & m_c & T_w \end{bmatrix}^T \qquad (5.51)$$

(3) $$x'' = [U_c \quad m_c \quad E_w]^T \qquad (5.52)$$

The resulting A-matrix for each of these models was derived analytically (Rassmussen, 2002) and are given in Equations 5.53 - 5.55. These are the numerical results given for a specific set of chosen dimensions.

$$A = \begin{bmatrix} -16.202 & -1711.7 & 2706.5 \\ -0.31485 & -33.263 & 52.593 \\ 0.0030427 & 0.2912 & -0.6006 \end{bmatrix} \qquad (5.53)$$

$$A' = \begin{bmatrix} -49.465 & 8.9885e6 & 2696.7 \\ 0 & 0 & 0 \\ 0.0087329 & -1534.9 & -0.6006 \end{bmatrix} \qquad (5.54)$$

$$A'' = \begin{bmatrix} -49.465 & -6170.6 & 0.50826 \\ 0 & 0 & 0 \\ 46.335 & 5930.1 & -0.6006 \end{bmatrix} \qquad (5.55)$$

To obtain a dimensionless representation, a simple method to do this is to select a new, problem-specific dimensional basis. Four dimensions are needed for length, mass, temperature, and time scales. They are:

(1) length – length of fluid flow in the gas cooler,

(2) mass – refrigerant mass inventory in the gas cooler,

(3) temperature – 273 K,

(4) time – refrigerant mass inventory divided by mass flow rate.

Each of the parameters representing these dimensional normalizations are evaluated at the steady state operating condition, which corresponds to the A/C system operating on a vehicle at highway driving conditions. The numerical values for this basis, as well as the resulting numerical values for the dimensionless states are given in Table 5.1.

Table 5.1: The dimensional basis for the gas cooler

		Pressure [kPa = kg/m/s^2]	Enthalpy [kJ/kg = m^2/s^2]	Temperature [K]	Mass [kg]	Energy [kJ = m^2*kg/s^2]
Length	2.285	-1	2			2
Mass	0.0423	1			1	1
Time	0.9646	-2	-2			-2
Temperature	273			1		
		1.989E+01	5.612E+03	2.730E+02	4.229E-02	2.373E+02

Using this basis, the dimensions of the system were transformed into pi-form via a similarity transformation given by Equations 5.56 - 5.58.

$$T = \begin{bmatrix} 19.8901 & 0 & 0 \\ 0 & 5611.50 & 0 \\ 0 & 0 & 273 \end{bmatrix} \tag{5.56}$$

$$T' = \begin{bmatrix} 19.8901 & 0 & 0 \\ 0 & 0.042288 & 0 \\ 0 & 0 & 273 \end{bmatrix} \tag{5.57}$$

$$T'' = \begin{bmatrix} 237.299 & 0 & 0 \\ 0 & 0.042288 & 0 \\ 0 & 0 & 237.299 \end{bmatrix} \tag{5.58}$$

This operation results in the matrices given in Equations 5.59 - 5.61 where the bar denotes the dimensionless system representation.

$$\overline{A} = T^{-1}AT = \begin{bmatrix} -16.202 & -4.8292e5 & 37147 \\ -0.001116 & -33.263 & 2.5587 \\ 0.00022168 & 5.9857 & -0.6006 \end{bmatrix} \tag{5.59}$$

$$\overline{A}' = T'^{-1}A'T' = \begin{bmatrix} -49.465 & 19108 & 37147 \\ 0 & 0 & 0 \\ 0.00063397 & -0.23683 & -0.6006 \end{bmatrix} \tag{5.60}$$

$$\overline{A}'' = T''^{-1}A''T'' = \begin{bmatrix} -49.465 & -1.0996 & 0.50826 \\ 0 & 0 & 0 \\ 46.335 & 1.0568 & -0.6006 \end{bmatrix} \tag{5.61}$$

Evaluating these system matrices for diagonal dominance using the measures outlined earlier in this chapter yields little useful information. None of the representations is clearly diagonally dominant, a statement that can be confirmed by inspection. The best representation is then determined by condition number. The condition number for each of the representations (ignoring the zero singular value) is given as 3465571, 651524, and 729.6 for the A, A', and A'' representations respectively. From this analysis, the third representation is chosen for model reduction. The necessary scaling matrix to obtain a balanced realization is calculated, and given as $S = diag([8 \quad 1 \quad 1])$. Thus the obvious choice of the state to be residualized corresponds to the first state of the A'' representation. Equation (5.3) reveals that this is the refrigerant energy.

5.3.1.1 Reduced Order Model

For discussion purposes, additional reduced-order models were calculated so that all three representations could be compared. These models are not described here, but their eigenvalues are calculated and compared to the eigenvalues of the full order model (Tables 5.2 - 5.4). From these tables it is clear that the first representation is a poor choice for model reduction, because the zero eigenvalue dynamic is not explicitly associated with any state. The reduced order models for both the second and third representation yield equivalent eigenvalues. This is due to the fact that the states of the second representation are simply constant multiples of the states of the third representation. For all representations, it is obvious that by residualizing the wall temperature/energy dynamics leads to the removal of the slowest eigenvalue. Thus we can conclude that for the gas cooler the refrigerant energy (equivalent to pressure) dynamics are fast, the wall temperature/energy dynamics are slow, and there is a pure integrator due to the conservation of mass requirements.

Table 5.2: Gas cooler eigenvalue comparison for reduced order models of A

	Eliminate: Pressure		Eliminate: Enthalpy		Eliminate: Wall Temp.	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-49.943					-10.254	79.5%
-0.123	-0.092	25.1%	-0.140	13.7%		
0	0	0.0%	0	0.0%	0	0.0%

Table 5.3: Gas cooler eigenvalue comparison for reduced-order models of A'

	Eliminate: Pressure		Eliminate: Wall Temp.	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-49.943			-10.254	79.5%
-0.123	-0.125	1.0%		
0	0	0.0%	0	0.0%

Table 5.4: Gas cooler eigenvalue comparison for reduced-order models of A''

	Eliminate: Refrigerant Energy		Eliminate: Wall Energy	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-49.943			-10.254	79.5%
-0.123	-0.125	1.0%		
0	0	0.0%	0	0.0%

The final reduced order model used is given (in dimensional form) in Equations 5.62 - 5.65. This reduced order model is a second order system with states defined as $x''_r = \begin{bmatrix} m_c & E_w \end{bmatrix}^T$.

$$A_r = \begin{bmatrix} 0 & 0 \\ 149.97 & -0.12451 \end{bmatrix} \quad (5.62)$$

$$B_r = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 52.894 & 105.72 & 0.081689 & 0.26625 & -5.2917 \end{bmatrix} \quad (5.63)$$

$$C_r = \begin{bmatrix} 1.8169e5 & 18.978 \\ -3439.5 & 0.73756 \\ 0 & 0.34685 \\ 0 & 0.079861 \\ 429.94 & 0.26266 \\ 1 & 0 \end{bmatrix} \quad (5.64)$$

$$D_r = \begin{bmatrix} 2108.4 & 4214 & 3.2562 & 0 & 0 \\ 81.943 & 163.78 & -0.87345 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.76975 & -0.98402 \\ 29.181 & 58.324 & -0.15346 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.65)$$

5.3.2 Evaporator Order Reduction

The second component in the AC system under consideration is the Evaporator. From Rasmussen's work on the subject (Rassmussen, 2002), there are again three possible choices of states for the evaporator. They are given as:

(1) $$x = \begin{bmatrix} L_1 & P_e & h_o & T_{w1} & T_{w2} \end{bmatrix}^T \quad (5.66)$$

(2) $$x' = \begin{bmatrix} L_1 & P_e & m_e & T_{w1} & T_{w2} \end{bmatrix}^T \quad (5.67)$$

(3) $$x'' = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 & \widetilde{m}_e & \widetilde{E}_{w1} & \widetilde{E}_{w2} \end{bmatrix}^T \quad (5.68)$$

The resulting A-matrix for each of the evaporator models are given in Equations 5.69 - 5.71. These are the numerical results given a particular dimension system described later in Table 5.5.

$$A = \begin{bmatrix} -1.3172 & 0.019652 & -0.0034846 & -1.8206 & 0.0077459 \\ -95.781 & -13.987 & -80.326 & 1080.3 & 178.56 \\ -556.93 & -1.0525 & -50.097 & -33.407 & 111.36 \\ -0.078365 & 0.016212 & 0 & -1.5469 & 0.0066648 \\ -15.566 & 0.23949 & 0.1724 & -21.378 & -0.7134 \end{bmatrix} \quad (5.69)$$

$$A' = \begin{bmatrix} -2.6367 & 0.017536 & 221.09 & -1.8206 & 0.0077459 \\ -30512 & -62.765 & 5.0966e6 & 1080.3 & 178.56 \\ 0 & 0 & 0 & 0 & 0 \\ -0.078365 & 0.016212 & 0 & -1.5469 & 0.0066648 \\ 49.712 & 0.34417 & -10938 & -21.378 & -0.7134 \end{bmatrix} \quad (5.70)$$

$$A'' = \begin{bmatrix} -12.999 & -17.483 & -2988.7 & 1.2049 & 0 \\ -1.2575 & -52.312 & -733.44 & 0 & 0.13311 \\ 0 & 0 & 0 & 0 & 0 \\ 14.535 & 21.746 & 3717.4 & -1.5469 & 0.0066648 \\ 16.045 & 189.24 & 3100.1 & 0.029154 & -0.80448 \end{bmatrix} \quad (5.71)$$

Again a dimensionless representation is desired, so a new, problem-specific dimensional basis was chosen. Four dimensions are needed for length, mass, temperature, and time scales. They are:

(1) length – length of fluid flow in the gas evaporator,

(2) mass – refrigerant mass inventory in the evaporator,

(3) temperature – 273 K,

(4) time – refrigerant mass inventory divided by mass flow rate.

Each of the parameters representing these dimensional normalizations are evaluated at the steady state operating condition, which corresponds to the A/C system operating on a vehicle at highway driving conditions. The numerical values for this basis, as well as the resulting numerical values for the dimensionless states are given in Table 5.5.

Table 5.5: New dimensional basis for the evaporator

		Length [m]	Pressure [kPa = kg/m/s^2]	Enthalpy [kJ/kg = m^2/s^2]	Temperature [K]	Mass [kg]	Energy [kJ = m^2*kg/s^2]
Length	1.859826	1	-1	2			2
Mass	0.0412		1			1	1
Time	0.9448		-2	-2			-2
Temperature	273				1		
		1.860E+00	2.482E+01	3.875E+03	2.730E+02	4.120E-02	1.596E+02

Using this new parameter basis to form a dimensionless system representation, the dimensionless system is obtained via a similarity transformation (Equations 5.72 - 5.74).

$$T = \begin{bmatrix} 1.85983 & 0 & 0 & 0 & 0 \\ 0 & 24.8150 & 0 & 0 & 0 \\ 0 & 0 & 3874.99 & 0 & 0 \\ 0 & 0 & 0 & 273 & 0 \\ 0 & 0 & 0 & 0 & 273 \end{bmatrix} \tag{5.72}$$

$$T' = \begin{bmatrix} 1.85983 & 0 & 0 & 0 & 0 \\ 0 & 24.8150 & 0 & 0 & 0 \\ 0 & 0 & 0.0411966 & 0 & 0 \\ 0 & 0 & 0 & 273 & 0 \\ 0 & 0 & 0 & 0 & 273 \end{bmatrix} \tag{5.73}$$

$$T'' = \begin{bmatrix} 159.636 & 0 & 0 & 0 & 0 \\ 0 & 159.636 & 0 & 0 & 0 \\ 0 & 0 & 0.0411966 & 0 & 0 \\ 0 & 0 & 0 & 159.636 & 0 \\ 0 & 0 & 0 & 0 & 159.636 \end{bmatrix} \quad (5.74)$$

This transform results in the system matrices given in Equations 5.75 - 5.77 where the bar denotes a dimensionless representation.

$$\bar{A} = T^{-1}AT = \begin{bmatrix} -1.30172 & 0.26221 & -7.2603 & -267.25 & 1.137 \\ -7.1785 & -13.987 & -12543 & 11884 & 1964.4 \\ -0.2673 & -0.00674 & -50.097 & -2.3536 & 7.8455 \\ -0.00053387 & 0.0014736 & 0 & -1.5469 & 0.0066648 \\ -0.13605 & 0.021769 & 2.447 & -21.378 & -0.7134 \end{bmatrix} \quad (5.75)$$

$$\bar{A}' = T'^{-1}A'T' = \begin{bmatrix} -2.6367 & 0.23398 & 4.8974 & -267.25 & 1.137 \\ -2286.8 & -62.765 & 8461 & 11884 & 1964.4 \\ 0 & 0 & 0 & 0 & 0 \\ -0.00053387 & 0.0014736 & 0 & -1.5469 & 0.0066648 \\ 0.33866 & 0.031285 & -1.6506 & -21.378 & -0.7134 \end{bmatrix} \quad (5.76)$$

$$\bar{A}'' = T''^{-1}A''T'' = \begin{bmatrix} -12.999 & -17.483 & -0.77127 & 1.2049 & 0 \\ -1.2575 & -52.312 & -0.18928 & 0 & 0.13311 \\ 0 & 0 & 0 & 0 & 0 \\ 14.535 & 21.746 & 0.95933 & -1.5469 & 0.0066648 \\ 16.045 & 189.24 & 0.80004 & 0.029154 & -0.80448 \end{bmatrix} \quad (5.77)$$

Examining these for diagonal dominance using the measures outlined earlier in this chapter again yields little useful information. Again the representations given are not diagonally dominant, which can be confirmed by inspection. The best representation is then determined by condition number. The condition number for each of the representations (ignoring the zero singular value) is given as 2659305, 2370524, and 6779.1 for the A, A', and A'' representations respectively. From this analysis, the third representation is chosen for model reduction. The necessary scaling matrix to obtain a balanced realization is calculated, and given as $S = diag([2 \quad 8 \quad 1 \quad 0.5 \quad 0.25])$. Thus the obvious choice of the states to be residualized is the second state of the A'' representation. Referring to Equation (5.6), this corresponds to the refrigerant energy in the second region of the evaporator. We also observe that the first state of the A'' representation might also be considered fast, and this state corresponds to the refrigerant energy in the first region. This is somewhat equivalent to residualizing the pressure and two-phase flow length in the second representation.

5.3.2.2 Reduced Order Model

For discussion purposes, reduced order models were again calculated for all three representations. The eigenvalues of these models (again, not all presented) were calculated and compared to the eigenvalues of the full order model (Tables 5.6 - 5.8).

Table 5.6: Evaporator eigenvalue comparison for reduced-order models of A

	Eliminate: Pressure		Eliminate: Pressure, Length		Eliminate: Pressure, Enthalpy	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-53.374	-43.033	19.4%				
-13.745						
-0.411	-0.375	8.9%	-0.914	122.3%	-0.336	18.3%
-0.132	-0.048	63.6%	-0.062	52.8%	-0.047	64.6%
0	0	0%	0	0%	0	0%

Table 5.7: Evaporator eigenvalue comparison for reduced-order models of A'

	Eliminate: Pressure		Eliminate: Pressure, Length	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-53.374				
-13.745	-11.622	15.4%		
-0.411	-0.409	0.6%	-0.375	8.8%
-0.132	-0.133	1.2%	-0.151	14.8%
0	0	0%	0	0%

Table 5.8: Evaporator eigenvalue comparison for reduced-order models of A''

	Eliminate: Refrig. Energy #2		Eliminate: Refrig. Energy #1, Refrig. Energy #2	
Full Order Eigenvalues	Reduced Order Eigenvalues	Percentage Error	Reduced Order Eigenvalues	Percentage Error
-53.374				
-13.745	-13.902	1.1%		
-0.411	-0.414	0.7%	-0.427	3.8%
-0.132	-0.132	0.2%	-0.141	7.4%
0	0	0%	0	0%

From these tables it should be clear that the first representation is a poor choice for model reduction, because the zero eigenvalue dynamic is not explicitly associated with any state. The reduced order models for both the second and third representation yield similar eigenvalues, but the third representation approximates the slow eigenvalues with the least error. From the previous gas cooler results, it is obvious that by residualizing the wall temperature/energy dynamics leads to the removal of the slowest eigenvalue. Thus we can conclude that for the evaporator the refrigerant energy (similar to pressure and two-phase flow length) dynamics are fast, the wall temperature/energy dynamics are slow, and there is a pure integrator in the model representing the conservation of mass.

The previous discussion indicated that either a 3rd or 4th order model may be chosen for the evaporator. The final reduced order models used are shown below. The 4th order model is given (in dimensional form) in Equations 5.78 - 5.81. This reduced order model is a fourth-order system with states defined as: $x''_{r,4} = \begin{bmatrix} \tilde{U}_1 & \tilde{m}_e & \tilde{E}_{w1} & \tilde{E}_{w2} \end{bmatrix}^T$.

$$A_{r,4} = \begin{bmatrix} -12.579 & -2743.6 & 1.2049 & -0.044488 \\ 0 & 0 & 0 & 0 \\ 14.012 & 3412.5 & -1.5469 & 0.061999 \\ 11.496 & 446.91 & 0.029154 & -0.32294 \end{bmatrix} \qquad (5.78)$$

$$B_{r,4} = \begin{bmatrix} -78.069 & 4.4896 & 0.043604 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -5.5843 & 0 & 0.12951 & 21.199 \\ 0 & -48.596 & 0 & 0.12951 & 16.386 \end{bmatrix} \qquad (5.79)$$

$$C_{r,4} = \begin{bmatrix} -0.70004 & 4.2265 & 0 & 6.8533e-5 \\ 400.05 & 97427 & 0 & 1.5798 \\ -22.141 & -2686.2 & 0 & 0.98528 \\ 0 & 0 & 0.46284 & 0 \\ -8.231 & 49.695 & 0 & 0.46364 \\ 0 & 0 & 0.28182 & 0.064427 \\ -8.1718 & -375.99 & 0 & 0.60761 \\ -12.553 & -1443 & 0 & 0.59031 \\ -3.8934e-5 & 0.99052 & 0 & -1.5375e-7 \end{bmatrix} \qquad (5.80)$$

$$D_{r,4} = \begin{bmatrix} 0 & -0.0069163 & 0 & 0 & 0 \\ 0 & -159.43 & 0 & 0 & 0 \\ 0 & -99.433 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.081322 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2519 & 16.784 \\ 0 & -61.32 & 0 & 0 & 0 \\ 0 & -59.574 & 0 & 0 & 0 \\ 0 & 1.5516e-5 & 0 & 0 & 0 \end{bmatrix} \qquad (5.81)$$

The 3rd order evaporator model is given (in dimensional form) in Equations 5.82 - 5.85. This reduced order model is a third order system with states defined as $x''_{r,3} = \begin{bmatrix} \tilde{m}_e & \tilde{E}_{w1} & \tilde{E}_{w2} \end{bmatrix}^T$.

$$A_{r,3} = \begin{bmatrix} 0 & 0 & 0 \\ 356.23 & -0.20472 & 0.012441 \\ -2060.4 & 1.1303 & -0.3636 \end{bmatrix} \qquad (5.82)$$

$$B_{r,3} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -86.967 & -0.58294 & 0.048574 & 0.12951 & 21.199 \\ -71.348 & -44.493 & 0.03985 & 0.12951 & 16.386 \end{bmatrix} \qquad (5.83)$$

$$C_{r,3} = \begin{bmatrix} 156.91 & -0.67053 & 0.0025444 \\ 10170 & 38.319 & 0.16491 \\ 2143.1 & -2.1208 & 1.0636 \\ 0 & 0.46284 & 0 \\ 1845 & -0.78841 & 0.49276 \\ 0 & 0.28182 & 0.064427 \\ 1406.4 & -0.78274 & 0.63651 \\ 1295 & -1.2024 & 0.63471 \\ 0.99901 & -3.7294e-6 & -1.605e-8 \end{bmatrix} \qquad (5.84)$$

$$D_{r,3} = \begin{bmatrix} 4.3448 & -0.25678 & -0.0024297 & 0 & 0 \\ -2482.9 & -16.643 & 1.3868 & 0 & 0 \\ 137.42 & -107.34 & -0.076752 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 51.086 & -3.0192 & -0.028533 & 0 & 0 \\ 0 & 0 & 0 & 0.2519 & 16.784 \\ 50.718 & -64.237 & -0.028328 & 0 & 0 \\ 77.91 & -64.054 & -0.43515 & 0 & 0 \\ 0.00024165 & 1.6197e-6 & -1.3497e-7 & 0 & 0 \end{bmatrix} \quad (5.85)$$

5.3.3 Internal Heat Exchanger Model Reduction

From previous analytical derivation, there is only one state-meaningful representation for the dynamics of a heat exchanger, and the A-matrix for this state-space representation are given by Equation 5.86. Because multiple representations do not need to be compared and the system representation is uniform, it is not necessary to obtain dimensionless model. However, the state(s) to be residualized have yet to be determined. The eigenvalues for A in Equation 5.86 are given in Equation 5.87. Because two of the eigenvalues are two orders of magnitude greater than the third, two of the states can be residualized.

$$A = \begin{bmatrix} -23.671 & 0 & 4.8072 \\ 0 & -134.21 & 88.852 \\ 0.24584 & 0.24584 & -0.49169 \end{bmatrix} \quad (5.86)$$

$$\lambda(A) = \begin{bmatrix} -134.375 \\ -23.722 \\ -0.277 \end{bmatrix} \quad (5.87)$$

By inspection, the ‘A’ matrix appears to be diagonally dominant. Using the definition of diagonal dominance presented earlier, it is confirmed that the mathematical requirements for diagonal dominance hold. Additionally, using the induced two-norm we find that the maximum singular value, 160.98, is the same order of magnitude as the largest eigenvalue, -134.38. Finally, the diagonal elements of the Relative Gain Array are relatively close to unity (Equation 5.88).

$$\Lambda(A) = \begin{bmatrix} 1.182 & 0 & -0.182 \\ 0 & 1.585 & -0.585 \\ -0.182 & -0.585 & 1.766 \end{bmatrix} \quad (5.88)$$

It should be clear that A is diagonally dominant. The logical choice of states to be residualized is then the 1st and 2nd states (refrigerant temperatures). We should retain the 3rd state (wall temperature). The resulting 1st order model has an eigenvalue of -0.278, which approximates the slow eigenvalue of the full-order system with 0.3% error. Again, the conclusion is reached that the refrigerant dynamics are much faster than the heat exchanger wall dynamics. The final reduced order model used is given in Equations 5.89 - 5.92.

$$A_r = [-0.27835] \qquad (5.89)$$

$$B_r = \begin{bmatrix} 12.864 & -38.651 & 0.0011452 & 0.00087845 & 0.034665 & 0.057 \end{bmatrix} \qquad (5.90)$$

$$C_r = \begin{bmatrix} 3.0489 \\ 1.5257 \\ 0.4115 \\ 1.3241 \end{bmatrix} \qquad (5.91)$$

$$D_r = \begin{bmatrix} 775.38 & 0 & -0.013118 & 0 & 0.62106 & 0 \\ 0 & -362.32 & 0 & -0.017126 & 0 & -0.15303 \\ 104.65 & 0 & 0.0031947 & 0 & 0.083821 & 0 \\ 0 & -314.44 & 0 & -0.0054408 & 0 & -0.1328 \end{bmatrix} \qquad (5.92)$$

5.3.4 Full-Order System After Model Reduction

The full order system model was analytically derived by Rasmussen (Rassmussen, 2002). Each of the reduced-order component models derived above are here combined to form a reduced-order system model. For this representation, only five outputs are considered:

$$y = \begin{bmatrix} T_{e,sh} & P_e & P_c & T_{e,ao} & T_{c,ao} \end{bmatrix}^T . \qquad (5.93)$$

If the 4th order model of the evaporator is used, the resulting system model is 6th order and presented in Equations 5.94 - 5.98.

$$A_{r,6} = \begin{bmatrix} -11.974 & -2272.9 & 1.2049 & -0.035315 & 0.0017847 & 0.088072 \\ -0.010924 & -4.6201 & 0 & -0.00010217 & 9.4659e-5 & 0.00012576 \\ 13.967 & 3399.3 & -1.5469 & 0.061606 & 0.0001621 & 0.0019738 \\ 11.1 & 332.19 & 0.029154 & -0.32636 & 0.0014107 & 0.017177 \\ -1.2815 & -792.61 & 0 & -0.023462 & -0.10501 & 0.12545 \\ -0.83431 & -129.9 & 0 & 0.061791 & 0.045393 & -0.32448 \end{bmatrix} \qquad (5.95)$$

$$B_{r,6} = \begin{bmatrix} -0.076305 & 0.00045552 & 0 & 0 & 0 & 0 \\ 0.0017762 & -2.6978e-5 & 0 & 0 & 0 & 0 \\ 6.2618e-5 & -0.00014587 & 0.12951 & 21.199 & 0 & 0 \\ 0.00054492 & -0.0012694 & 0.12951 & 16.386 & 0 & 0 \\ 0.19144 & 0.0004088 & 0 & 0 & 0.26625 & -5.2917 \\ 0.039912 & -0.00075733 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.96)$$

$$C_{r,6} = \begin{bmatrix} -13.037 & -4583.6 & 0 & 0.58612 & 0.0017293 & 0.021057 \\ 398.76 & 97050 & 0 & 1.5686 & 0.0046281 & 0.056352 \\ -51.081 & -2.073e5 & 0 & -0.93521 & 19.755 & 5.0006 \\ 0 & 0 & 0.28182 & 0.064427 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.079861 & 0 \end{bmatrix} \quad (5.97)$$

$$D_{r,6} = \begin{bmatrix} 0.00066801 & -0.0015562 & 0 & 0 & 0 & 0 \\ 0.0017877 & -0.0041646 & 0 & 0 & 0 & 0 \\ 7.6308 & 0.016295 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2519 & 16.784 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.76975 & -0.98402 \end{bmatrix} \quad (5.98)$$

If the 3rd order model of the evaporator is used, the resulting system model is 5th order and presented in Equations 5.99 - 5.102. Note that the reduced-order system has one redundant state from the conservation of energy in both the evaporator and gas cooler, evidenced by the zero eigenvalue. Thus the correct representation of the reduced-order system is found by combining the reduced-order component models and removing the redundant state.

$$A_{r,5} = \begin{bmatrix} -2.5464 & -0.0010992 & -6.995e-5 & 9.303e-5 & 4.5413e-5 \\ 747.96 & -0.14146 & 0.020411 & 0.002244 & 0.10471 \\ -1775 & 1.1462 & -.35911 & 0.0030653 & 0.098827 \\ -549.35 & -0.12895 & -0.019682 & -0.1052 & 0.11603 \\ 28.48 & -0.083953 & 0.064252 & 0.045268 & -0.33062 \end{bmatrix} \quad (5.99)$$

$$B_{r,5} = \begin{bmatrix} 0.0018458 & -2.7394e-5 & 0 & 0 & 0 & 0 \\ -0.088945 & 0.00038548 & 0.12951 & 21.199 & 0 & 0 \\ -0.070196 & -0.00084712 & 0.12951 & 16.386 & 0 & 0 \\ 0.1996 & 0.00036005 & 0 & 0 & 0.26625 & -5.2917 \\ 0.045229 & -0.00078907 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.100)$$

$$C_{r,5} = \begin{bmatrix} 891.31 & -1.3119 & 0.62457 & -0.00021397 & -0.074841 \\ 21354 & 40.125 & 0.39245 & 0.064065 & 2.9894 \\ -1.9761e5 & -5.1401 & -0.78455 & 19.747 & 4.6249 \\ 0 & 0.28182 & 0.064427 & 0 & 0 \\ 0 & 0 & 0 & 0.079861 & 0 \end{bmatrix} \quad (5.101)$$

$$D_{r,5} = \begin{bmatrix} 0.083753 & -0.0020522 & 0 & 0 & 0 & 0 \\ -2.5394 & 0.011005 & 0 & 0 & 0 & 0 \\ 7.9563 & 0.014352 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2159 & 16.784 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.76975 & -0.98402 \end{bmatrix} \quad (5.102)$$

The eigenvalues of the 5th-order system model approximates the eigenvalues of the full-order model within 11% error (Table 5.9). The 6th order model approximates the eigenvalues of the full-order model within 8%.

Table 5.9: Comparison of system eigenvalues: full-order and reduced-order models

Full Order Eigenvalues	-124.02	-54.165	-49.608	-28.09	-14.598	-1.995	-0.472 ± 0.233i	-0.175	-0.0607	0
Reduced Order Eigenvalues	Eliminated	Eliminated	Eliminated	Eliminated	-15.648	-2.060	-0.475 ± 0.232i	-0.177	-0.0613	0
Percentage Error					**7.2%**	**3.3%**	**0.4%**	**1.4%**	**0.9%**	**0.0%**
Reduced Order Eigenvalues	Eliminated	Eliminated	Eliminated	Eliminated	Eliminated	-2.202	-0.518 ± 0.238i	-0.182	-0.0628	0
Percentage Error						**10.4%**	**8.3%**	**3.9%**	**3.4%**	**0.0%**

5.3.5 Verification of the Model Reduction

To verify that the reduced order model approximations are sufficient, simulation results for both the 6th and 5th order system models are compared to the original nonlinear and linearized models, as well as data. Figures 5.3 - 5.8 show that residualizing the fast states has negligible impact on the transient response of the system. In fact, the simulation results from both reduced order models are indistinguishable from the full order linearized model (11th order).

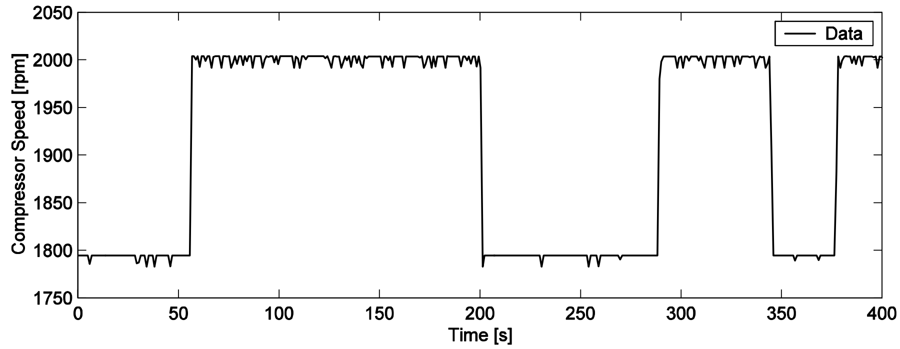


Figure 5.3: Compressor speed changes

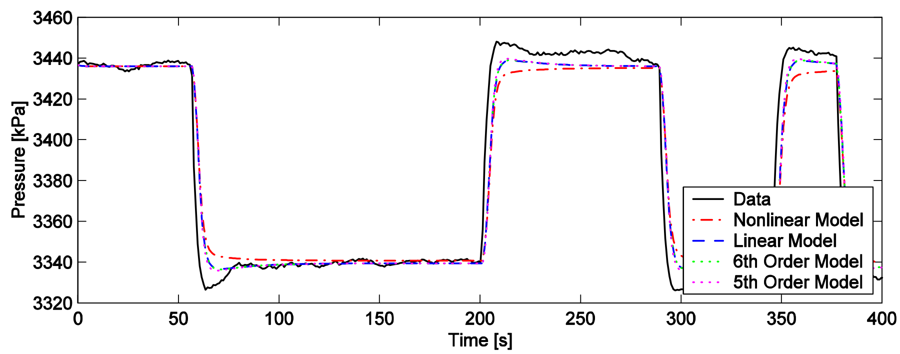


Figure 5.4: Evaporator pres for changes in compressor speed

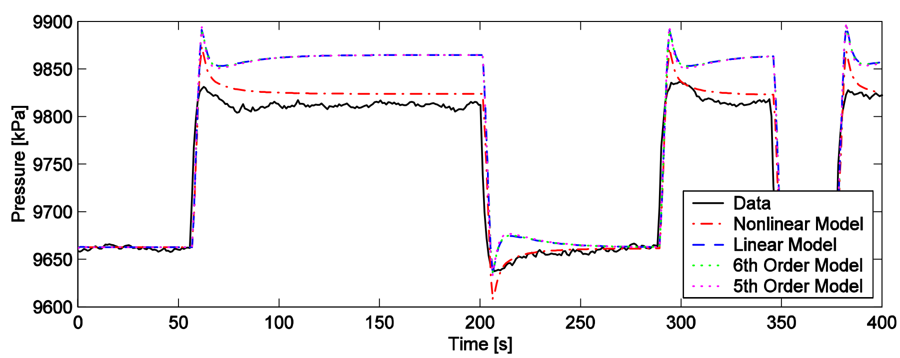


Figure 5.5: Gas cooler pressure for step changes in compressor speed

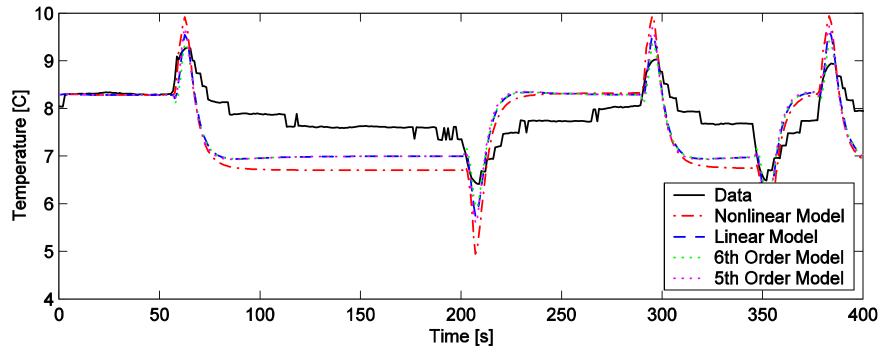


Figure 5.6: Evaporator superheat for step changes in compressor speed

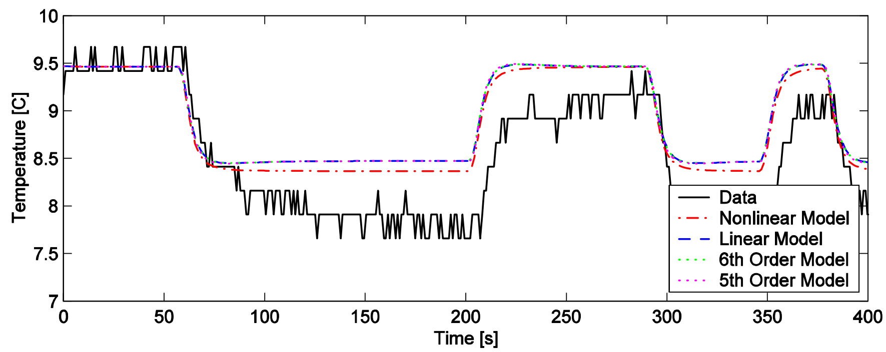


Figure 5.7: Evaporator exit air temperature for step changes in compressor speed

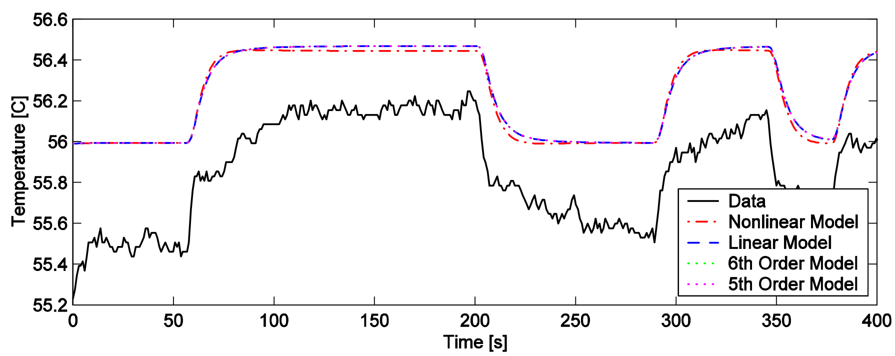


Figure 5.8: Gas cooler exit air temperature for step changes in compressor speed

While attempting to discover the dominant dynamics of the system, several alternative model reduction attempts were also made with mixed success based on physical parameters of the system model.

The wall temperature/energy dynamics have been shown to be the dominant dynamics of the system, along with the location of refrigerant mass. A logical step for reducing the system order further is to simplify the wall temperature assumptions by considering a single uniform wall temperature rather than separate wall temperatures for each region. The principle difficulty with this approach is with the calculated initial conditions. Other researches, namely Frank (Frank, 1978) have shown that singularly perturbed systems have discontinuous sensitivity functions with respect to initial conditions. This arises because the reduced-order model will have different initial conditions for the fast states in algebraic form than they will in differential form. One must use Tikhonov's Theorem (again, see (Frank, 1978) for details) to determine appropriate initial conditions on the reduced-order model.

If one ignores the sensitivity theory and attempts to use a finer mesh on the numerical calculation, one would then be using the full-order system to estimate by calculation the required initial conditions on the reduced-order model. Let us ignore the fact that these are numerically incompatible because the derivation of the singular dynamic system predicts a disparity between full-order and reduced-order initial conditions. Instead, given measured data and component parameters, the initial conditions for the dynamic state variables can be calculated. This includes the lumped wall temperatures and the effective length of two-phase flow. When a uniform wall temperature is assumed, the resulting initial condition can be drastically different from that calculated assuming separate wall temperatures. This difference will greatly affect the transient response and incorrect simulation results will be obtained.

Experience has shown that residualizing the wall temperature state in the gas cooler model leads to gross errors in the prediction of gas cooler exit air temperature. However, the effects on the other system outputs appear to be limited. Thus if gas cooler air temperature is not a variable of concern, a possibility exists of reducing the order of the system model further. Again, more research is needed before a recommendation can be made.

In summary, an 11^{th} order dynamic model for a transcritical air-conditioning system has been reduced to a 5^{th} order dynamic model without considerable loss in model accuracy.

Experience has shown that further reduction may be possible. The common model resulting from the PDE derivation was shown to be less desirable for model reduction. The dominant dynamics of the system were identified to be the wall temperature/energy dynamics and the location of refrigerant mass. The refrigerant energy dynamics were shown to be faster than the dominant dynamics by an order of magnitude, and could thus be residualized without notable loss of model accuracy.

A dimensional analysis was central to preserving the physical meaning of the system yet allowing the model reduction to produce valid results. The necessity of a dimensional analysis arose from the need to compare dimensionally inconsistent state representations. The beneficial results of this approach should be apparent from the results obtained.

5.4 Gain-Scheduling Simplification via Dimensionless Parameters

The previous section dealt with parameter groupings that are conducive to identification of system dynamics. A similar advantage of selective parameter groupings is obtained when considering systems whose parameters are already known. Specifically, careful selection of pi groupings can have significant advantages for both the implementation and the design of a control law.

5.4.1 Accidental Gain Scheduling – A Vehicle Example

The first realization that dimensional analysis would be very useful in a gain-scheduling aspect arose from more of an accidental discovery than intentional effort. In the previous chapter, we introduced the governing parameters for a model for the planar motions of a vehicle on the road. To reintroduce the problem, the governing parameters are:

Variable	Symbol	Value	Dimension
length from front axle to C.G.	a	0.1604	m
length from rear axle to C.G.	b	0.2048	m
front tire forces produced per unit slip	$C_{\alpha f}$	64.9	$m \cdot kg \cdot s^{-2}$
rear tire forces produced per unit slip	$C_{\alpha r}$	75.5	$m \cdot kg \cdot s^{-2}$
z-axis moment of inertia	I_z	0.1829	$m^2 \cdot kg$
mass of the vehicle	m	5.4510	kg
length of the vehicle	L	0.3652	m
speed of the vehicle	U	2.95	$m \cdot s^{-1}$

and the resulting pi-values are:

$$\pi_1 = \frac{a}{L}, \pi_2 = \frac{b}{L}, \pi_3 = \frac{C_{\alpha f} \cdot L}{mU^2}, \pi_4 = \frac{C_{\alpha r} \cdot L}{mU^2}, \pi_5 = \frac{I_z}{mL^2} \qquad (5.103)$$

As mentioned earlier, pi-parameters 1, 2, and 5 are geometric relationships that remain static, while parameters 3 and 4 vary significantly with vehicle velocity, U, and the cornering stiffness (a measure of road friction), $C_{\alpha f}$. Assuming the state vector is defined as [lateral position, lateral velocity, yaw angle, yaw rate],

$$\mathbf{x} \equiv \begin{bmatrix} y & \frac{dy}{dt} & \psi & \frac{d\psi}{dt} \end{bmatrix}^T \qquad (5.104)$$

and front steering input, $\mathbf{u} \equiv \begin{bmatrix} \delta_f \end{bmatrix}^T$, as the sole control channel, the governing dynamics in the traditional form are as follows: or:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t) \end{aligned} \qquad (5.105)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{f_1}{mU} & \frac{f_1}{m} & \frac{-f_2}{mU} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-f_2}{I_z \cdot U} & \frac{f_2}{I_z} & \frac{-f_3}{I_z \cdot U} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ \frac{C_{\alpha f}}{m} \\ 0 \\ \frac{a \cdot C_{\alpha f}}{I_z} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix} \qquad (5.106)$$

with:

$$f_1 = C_{\alpha f} + C_{\alpha r}, f_2 = a \cdot C_{\alpha f} - b \cdot C_{\alpha r}, f_3 = a^2 \cdot C_{\alpha f} + b^2 \cdot C_{\alpha r} \qquad (5.107)$$

Assuming nominal values given in the table above, we now investigate the *parameter* root locus with respect to large velocity or low friction.

Figure 5.9: Two destabilizing driving scenarios: high-speeds and low-friction

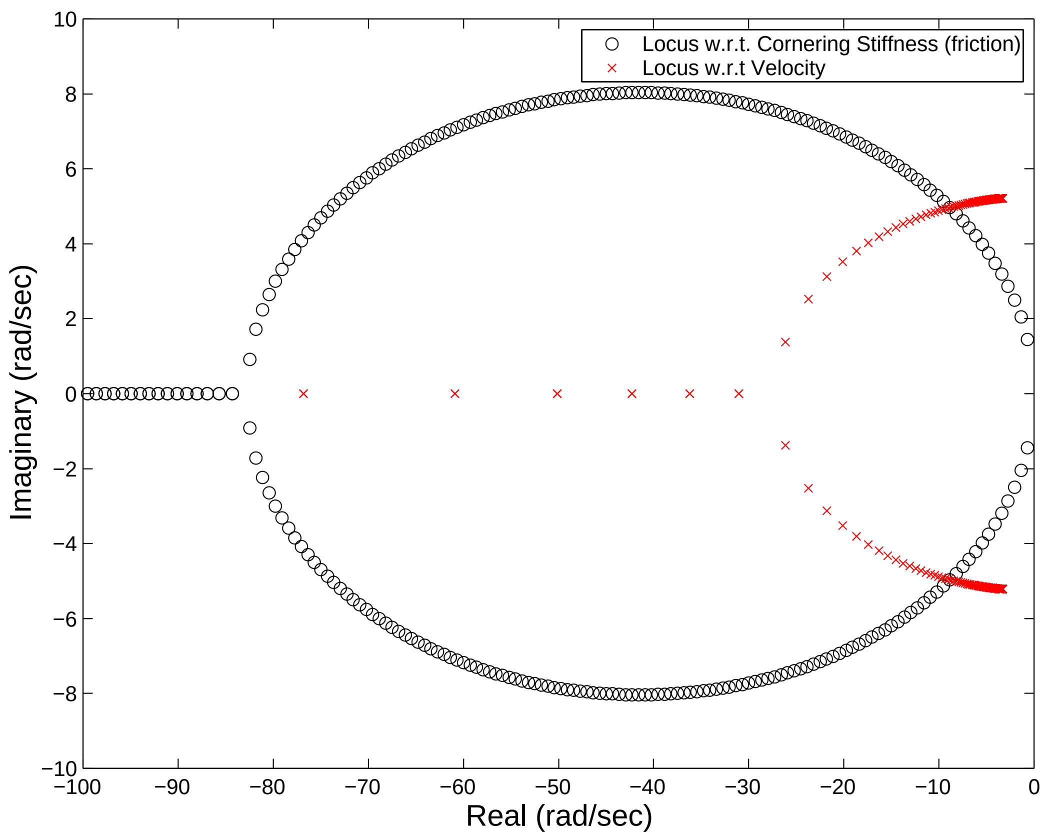


Figure 5.10: Parameter root locus w.r.t. cornering stiffness (friction) and velocity

For yaw stability, we investigate the pole locations (eigenvalues of A) ignoring the double integrator. The locus is shown in Figure 5.10, and the result is anticipated: for very low cornering stiffnesses (similar to low friction conditions), the system acts as four free integrators because the vehicle has two unconstrained kinematic modes: rotation and translation. At very high speeds, the system acts increasingly like a very oscillatory system, a result that agrees well with classical analysis and experimental vehicle testing.

If we consider, however, the *normalized form* of the above model, the system matrices become:

$$\begin{aligned} \dot{\mathbf{x}}^* &= \mathbf{A}^*\mathbf{x}^* + \mathbf{B}^*\mathbf{u}^* \\ \mathbf{y}^* &= \mathbf{C}^*\mathbf{x}^* + \mathbf{D}^*\mathbf{u}^* \end{aligned} \tag{5.108}$$

With:

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -p_1 & p_1 & -p_2 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-p_2}{\pi_5} & \frac{p_2}{\pi_5} & \frac{-p_3}{\pi_5} \end{bmatrix}, \mathbf{B}^* = \begin{bmatrix} 0 \\ \pi_3 \\ 0 \\ \frac{\pi_1 \cdot \pi_3}{\pi_5} \end{bmatrix}, \mathbf{C}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \ \mathbf{D}^* = [0] \tag{5.109}$$

$$p_1 = \pi_3 + \pi_4, p_2 = \pi_1\pi_3 - \pi_2\pi_4, p_3 = \pi_1^2\pi_3 + \pi_2^2\pi_4 \tag{5.110}$$

If, prior to the above normalization, we again perform a set of parameter variations and then measure the two different root loci of the *normalized* system under the constraint $C_{\alpha f} = k \cdot C_{\alpha r}, k\ fixed$, we obtain Figure 5.11.

Clearly, both parameter variations have the same root locus in the *normalized* system equations. That the normalized system has equivalent sensitivity to the two effects should be apparent from Equation 5.1, since both terms $C_{\alpha f}$ and U enter the equations only through the same dimensionless grouping. From the above plot, we may conclude that only *one* gain scheduling routine is then needed to schedule with respect to both parameters! We now generalize this result to arbitrary systems.

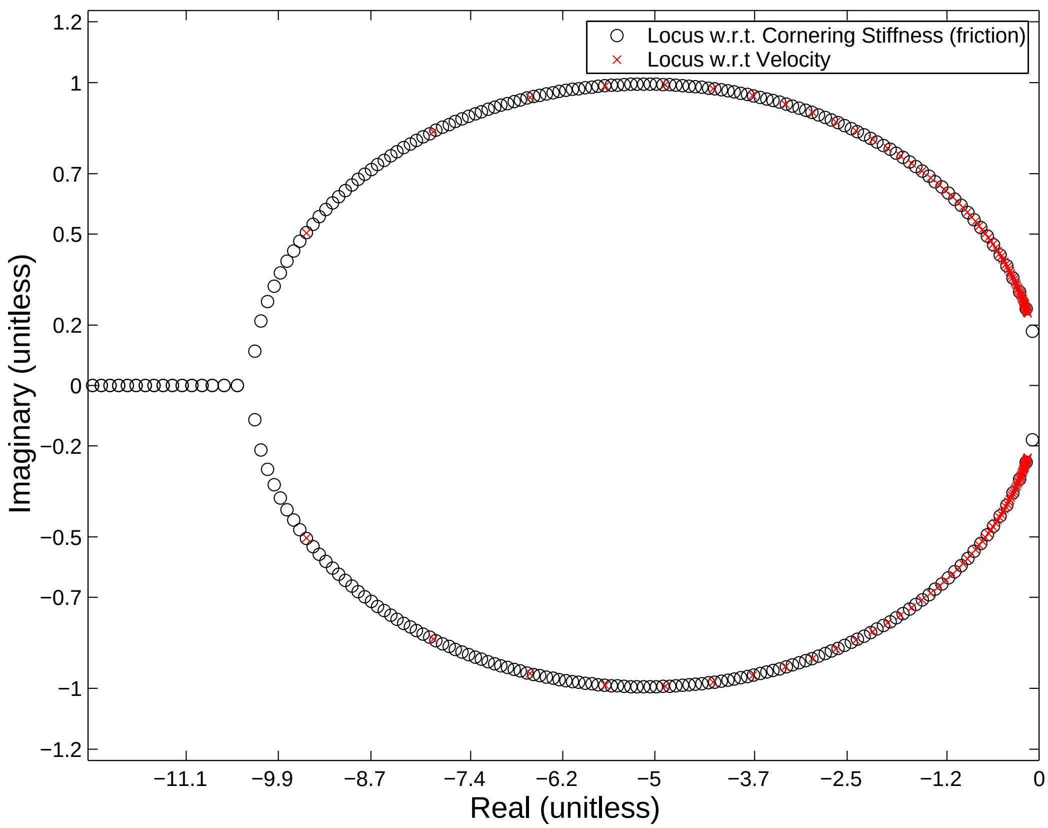


Figure 5.11: Dimensionless form of the parameter root locus

5.4.2 Generalized Unified Gain-Scheduling By Pi Parameterization

Theorem of Equivalent Gain Scheduling: (developed by Sean Brennan and Andrew Alleyne during the course of this thesis) *If the dimensional basis vectors of n+1 scheduling parameters in a system equation have rank n, then as many as n+1 varying system parameters can be equivalently represented by a single varying pi-parameter within an equivalent, dimensionless form of the system equation.*

Proof: (direct proof) If the dimensional basis vectors have rank n, then some subset n of the n+1 parameters can be selected as a new dimensional basis. The dimensional basis of remaining 1 parameter can then be expressed in dimensionless form in the new dimensional basis.

Example: Consider a modification of the previous vehicle problem where only the following parameters are relevant:

Variable	Symbol	Value	Dimension
front tire forces produced per unit slip	$C_{\alpha f}$	64.9	$\mathrm{m \cdot kg \cdot s^{-2}}$
mass of the vehicle	m	5.4510	kg
length of the vehicle	L	0.3652	m
speed of the vehicle	U	2.95	$\mathrm{m \cdot s^{-1}}$

There are four parameters that span 3 dimensions, and a simple analysis shows that the 4 dimensional basis vectors are of rank 3 and that *any* combination of the three vectors can be used as a rank 3 dimensional basis. Therefore, any system dynamics that is scheduled with respect to any or *all* of the above parameters can be unified by scheduling with respect to a dimensionless parameter formed from the above four groups. It is left to the curious reader to show that the parameter root locus with respect to mass and length also yields the same parameter locus as presented earlier, assuming geometric similarity constraints are satisfied.

5.4.3 Intentional Gain-Scheduling: Gantry Example

An interesting system where an intentional method of gain scheduling with respect to multiple parameters would be beneficial is a typical gantry system. A gantry is shown in Figure 5.12:

For the typical motion of a gantry (loading, lifting, unloading, moving), we note that the mass of the cargo canister as well as the arm length may be changing. We therefore would like to design a control law that schedules with respect to both of these parameters simultaneously.

The equations of motion are given below, and are derived in Franklin, Powell, and Emami-Naeini as a standard student example (Franklin, Powell, and Emami-Naeini, 2002):

$$\begin{gathered}\left(I+m_p \cdot l^2\right)\cdot\ddot{\theta}+m_p \cdot g \cdot l \cdot \sin(\theta) = -m_p \cdot l \cdot \ddot{x}\cdot\cos(\theta)\\ \left(m_t+m_p\right)\cdot\ddot{x}+b\cdot\dot{x}+m_p \cdot l\cdot\ddot{\theta}\cdot\cos(\theta)-m_p\cdot l\cdot\dot{\theta}^2\cdot\sin(\theta)=u\end{gathered} \tag{5.111}$$

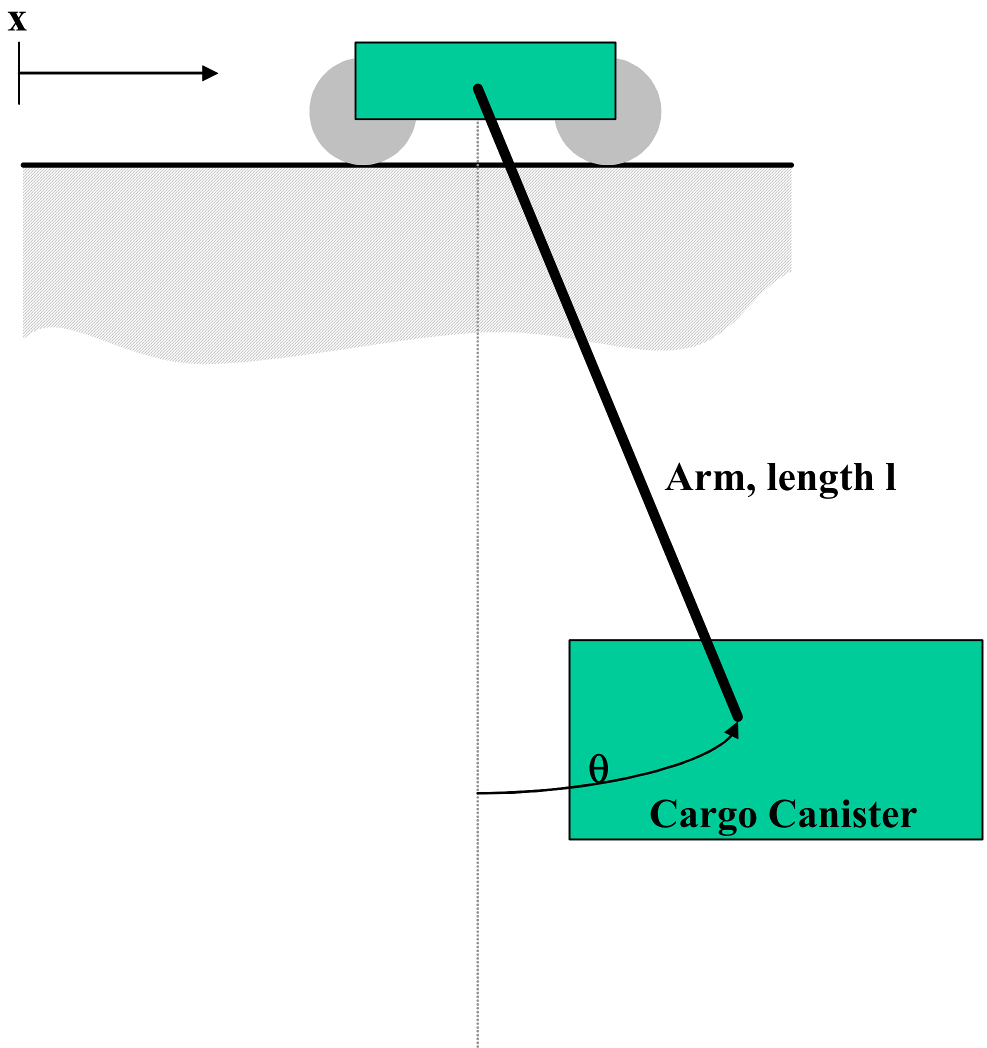


Figure 5.12: Diagram of a gantry system

The meaning of the equation parameters and their physical dimensions are defined as follows:

Variable	Symbol	Dimension
Mass of the loaded pendulum	m_p	kg
Mass of the trolley	m_t	kg
Length of pendulum arm	l	m
Viscous damping on the trolley	b	$kg \cdot s^{-1}$
Gravitational acceleration	g	$m \cdot s^{-2}$
Moment of inertia of arm	I	$kg \cdot m^2$
Angle of the arm w.r.t. vertical	θ	1
Position of arm horizontally	x	m
Control force on arm	u	$kg \cdot m \cdot s^{-2}$

We may simplify the problem somewhat by making the following general assumptions:

(1) The mass of the trolley is negligible with respect to the load it is carrying

(2) The moment of inertia of the rope or cable holding the load is negligible

The equations of motion become:

$$\begin{aligned} m_p \cdot l^2 \cdot \ddot{\theta} + m_p \cdot g \cdot l \cdot \sin(\theta) = -m_p \cdot l \cdot \ddot{x} \cdot \cos(\theta) \\ m_p \cdot \ddot{x} + b \cdot \dot{x} + m_p \cdot l \cdot \ddot{\theta} \cdot \cos(\theta) - m_p \cdot l \cdot \dot{\theta}^2 \cdot \sin(\theta) = u \end{aligned} \qquad (5.112)$$

We now wish to form a dimensionless model representation where the two scheduling parameters of load mass, m_p, and pendulum length, l, are be coupled by a single parameter. The previous theorem indicated that this should be possible, and we demonstrate that this is indeed the case by a direct pi-parameterization of the governing model. Specifically, we note that the spatial derivatives can be rewritten in the form:

$$\begin{aligned} & t \equiv S \cdot \bar{t} \\ & \dot{\theta} \equiv \frac{1}{S}\bar{\dot{\theta}} \Rightarrow \ddot{\theta} \equiv \frac{1}{S^2}\bar{\ddot{\theta}} \\ & \dot{x} \equiv \frac{L}{S}\bar{\dot{x}} \Rightarrow \ddot{x} \equiv \frac{L}{S^2}\bar{\ddot{x}} \\ & u \equiv \frac{L \cdot M}{S^2} \cdot \bar{u} \end{aligned} \qquad (5.113)$$

where L, M and S are spatial, mass, and time scaling factors. To form a dimensionless representation, we fix the dimensional scaling factors from parameters internal to the problem representation (versus meters, kilograms, and seconds, which are defined external to the problem). For this instance, we choose:

$$\begin{aligned} S &= \sqrt{\frac{l}{g}} \\ L &= l \\ M &= m_p \end{aligned} \qquad (5.114)$$

Substitution of Equations (5.3) and (5.6) into the governing dynamics of Equation 5.4 yield the following form:

$$
\begin{aligned}
m_p \cdot g \cdot l \cdot \ddot{\bar{\theta}} + m_p \cdot g \cdot l \cdot \sin\left(\bar{\theta}\right) &= -m_p \cdot g \cdot l \cdot \ddot{\bar{x}} \cdot \cos\left(\theta\right) \\
m_p \cdot g \cdot \ddot{\bar{x}} + b \cdot \sqrt{g \cdot l} \cdot \dot{\bar{x}} + m_p \cdot g \cdot \ddot{\bar{\theta}} \cdot \cos\left(\bar{\theta}\right) - m_p \cdot g \cdot \dot{\bar{\theta}}^2 \cdot \sin\left(\bar{\theta}\right) &= m_p \cdot g \cdot \bar{u}
\end{aligned} \qquad (5.115)
$$

Dividing through by the first term gives:

$$
\begin{aligned}
\ddot{\bar{\theta}} + \sin\left(\bar{\theta}\right) &= -\ddot{\bar{x}} \cdot \cos\left(\theta\right) \\
\ddot{\bar{x}} + \frac{b}{m_p} \cdot \sqrt{\frac{l}{g}} \cdot \dot{\bar{x}} + \ddot{\bar{\theta}} \cdot \cos\left(\bar{\theta}\right) - \dot{\bar{\theta}}^2 \cdot \sin\left(\bar{\theta}\right) &= \bar{u}
\end{aligned} \qquad (5.116)
$$

We notice that the final system contains only one parameter, namely:

$$
\pi_1 = \frac{b}{m_p} \cdot \sqrt{\frac{l}{g}} \qquad (5.117)
$$

In verification of the above (unproved) theorem, the problem can be parameterized such that both scheduling variables appear in a single system parameter. For this specific problem, the scheduling parameter is the *only* parameter of the system and it only appears once! Thus, a controller design (linear or nonlinear) should be relatively straightforward and easy to parameterize with respect to this single variable.

5.5 Robust Control by Parametric Uncertainty Descriptions, an LMI Approach

To illustrate the utility of using dimensional analysis, a robust control problem is considered focusing on vehicle dynamics where the model uncertainty is framed as parametric variation. The intent of this section is to illustrate many of the points previously discussed: that pi-parameterization provides an intuitive and useful measure of the nominal plant and the unmodeled dynamics. Specifically, a temporal and spatial re-parameterization of the linear vehicle Bicycle Model is presented utilizing pi-groups rather than traditional parametric representations. Previous discussion of the pi-groups using compiled data (Chapter 4) revealed a normal distribution in each of the parameters. Additionally, the pi-parameters themselves are highly interrelated, and exist about a line through pi-space. As mentioned earlier, the normal distribution suggests numerical pi values for an 'average' vehicle and maximum perturbations about the average. A state-feedback controller is designed in this section utilizing the pi-space line and the expected pi-perturbations to robustly stabilize all vehicles encompassed by the normal distribution of vehicle parameters shown earlier. Experimental verification is then obtained using a scaled vehicle.

5.5.1 Literature Review of Publications Related to Robust Vehicle Control

The field of Robust Control made large advances in the 1980's and a framework for formally dealing with system uncertainty is now fairly well understood (Zhou, Doyle, and Glover, 1996). However, there has been no work done to utilize a specific dimensionless structure to the problem in order to define plant deviations. An example of a system where a specific structure could be exploited is that of vehicle control. With extensive previous work that has been done on Automated Highway systems or Intelligent Vehicles (Shladover, 1995), it has been found that repeated manipulation of the various controllers is necessary to achieve adequate performance for different vehicles. This re-calibration of vehicle controllers is expected since the actual vehicle plant is changing from vehicle-to-vehicle.

In this example, we consider methods to design a controller that is robust to vehicle-to-vehicle variations, based on perturbations of a system plant around some 'average' vehicle. Work on vehicles of different sizes (Brennan, 1999) has indicated that it can be difficult to predict the variation magnitudes in the vehicle system dynamics before a new vehicle is built. An additional but related problem is that the notion of an 'average' vehicle is unclear. Average parameters are highly desirable in research vehicles to ensure appropriate controller development, but a methodology to compare research vehicles at different institutions based on their physical parameter dimensions has not yet been formalized in the field of control. Therefore there are two goals of this study: first, to develop a numerically appropriate framework that allows parameter-based comparisons between vehicles, and second to obtain a controller that is robust to vehicle-to-vehicle parameter variation.

What the control architecture often overlooks is the fact that single physical parameters do not usually change on a system independent of other parameters. For instance, if the mass of a vehicle is increased, the moment of inertia will increase as well. Although significant control theory has been developed to describe the *dynamic* relationships of a dynamical class of systems, this theory usually doesn't incorporate *parametric* trends introduced by the physical design constraints of similar systems. In vehicle systems for instance, this paper shows later that parameter interdependence can be described quite well by a line through parameter space.

In control theory, plant variations or uncertainty are usually resolved using two methods: robust control or adaptive control. Robust control seeks to design a controller unresponsive to variations, while adaptive control seeks to identify the model parameters and include the identified variations into the control of the plant. Examples of adaptive approaches applied to vehicle control include neural networks to identify and adapt to road friction changes (Shiotsuka, Nagamatsu, and Yoshida, 1993). Robust control to address road friction variation and velocity variation are presented in (Tagawa, *et al.*, 1996), and road-friction uncertainty robustness is presented in (Ono, *et al.*, 1994). Naturally, mixed approaches can be implemented; for instance in (Horiuchi, Yuhara, and Takei, 1996), where adaptive control is used to identify the vehicle model and robust control is used to stabilize the vehicle in the presence of expected model perturbations and disturbances. In the context of this research, the focus will be on categorizing the uncertainty within a non-dimensional framework. Therefore, the robust approach will be the tool used to consider model variability.

The pi distributions presented earlier provide a clear numerical measure of an 'average' vehicle as well as an expected level of vehicle-to-vehicle variability. In this section, a single controller is designed for the 'average' vehicle represented in the pi distributions. A robustness constraint is added that the same controller must also be robust to the vehicle-to-vehicle variations observed in these distributions. Thus, the resulting controller should therefore be able to control any passenger road vehicle. We are treating static vehicle-to-vehicle parameter variations as parameter perturbation of a single 'average' vehicle. The resulting controller could then be used as a starting controller for tuning a specific controller design, and only the knowledge of a single pi parameter is required. The reasoning and choice of the single pi parameter are discussed shortly.

Before seeking an 'all-vehicle-capable' controller, it should be mentioned that designing a controller suitable for many different vehicles is analogous to designing a controller for a single vehicle under many different operating conditions. Of course, we are assuming that the operating conditions are not rapidly changing. Single-vehicle robustness to operating conditions is a well studied problem, and it is known that obtaining a single fixed gain controller that stabilizes over a wide range of velocities and road surface conditions is challenging (Guldner, Tan, and Patwardhan, 1996). For the control of vehicle planar dynamics, if a very large range of parameter perturbations is to be considered it is quite possible that a single fixed-gain controller does not exist. In practice, gain scheduling of the controller is often required with respect to speed or road conditions.

5.5.2 Definition of the Nominal Plant

From the parameter distributions, the average pi parameters were obtained by fitting a normal curve to the above plots. However, during the design of this controller, the NHTSA database was unavailable. Hence, many outliers remain in the data and the dataset is not as extensive. As a consequence, these parameters, referred to as 'old', are slightly different from those reported earlier. For purposes of this calculation, the speed is fixed at a constant 32 m/s, or about 70 mph.

$$OLD \Rightarrow \bar{\pi}_1 = 0.4390, \bar{\pi}_3 = 0.1450, \bar{\pi}_4 = 0.1521 \cdot \bar{\pi}_3, \bar{\pi}_5 = 0.2414 \quad (5.118)$$

$$NEW \Rightarrow \bar{\pi}_1 = 0.4431, \bar{\pi}_3 = 0.1422, \bar{\pi}_4 = 0.1562, \bar{\pi}_5 = 0.2510 \quad (5.119)$$

A comparison of the 'old' and 'new' average parameter values shows an important design 'robustness' that is a simple result of the central limit theorem. Even with a significant refinement in the vehicle database, the averages do not change significantly. Thus, the definition of the 'nominal' plant is obviously well-defined, and the consequent controller design should be quite general. For this analysis, the defining length, mass, and velocity were chosen from the Uberquad design of 2000-2001:

$$m = 6.02[kg], L = 0.3585[m], U = 2.0[m/s] \quad (5.120)$$

From these values, 'average' vehicle parameters were calculated by using the formulas for each pi parameter:

$$
\begin{aligned}
\bar{a} = \bar{\pi}_1 \cdot L \quad &= 0.1574[m] \\
\bar{b} = \bar{\pi}_2 \cdot L \quad &= 0.2011[m] \\
\bar{C}\alpha f = \bar{\pi}_3 \cdot m \cdot U^2 / L &= 9.7392[N/rad] \\
\bar{C}\alpha r = \bar{\pi}_4 \cdot m \cdot U^2 / L &= 10.2183[N/rad] \\
\bar{I}z = \bar{\pi}_5 \cdot m \cdot L^2 \quad &= 0.1868[N/rad]
\end{aligned} \tag{5.121}
$$

The vehicle transfer function (see Chapter 2 for derivation) becomes, upon numerical substitution of the average values:

$$
\frac{\bar{y}(s)}{\delta_f(s)} = \frac{1}{s^2} \cdot \frac{1.62 \cdot s^2 + 3.19 \cdot s + 31.73}{s^2 + 3.41 \cdot s + 5.64} \tag{5.122}
$$

The state-space representation (Chapter 2) becomes numerically:

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.658 & 3.315 & 0.0434 \\ 0 & 0 & 0 & 1 \\ 0 & 1.398 & -2.797 & -1.752 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1.618 \\ 0 \\ 8.206 \end{bmatrix} \tag{5.123}
$$

With the substitution of the average pi parameters, the dimensionless form of the state-space equations (again, see Appendix) gives:

$$
A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.2971 & 0.2971 & 0.0217 \\ 0 & 0 & 0 & 1 \\ 0 & 0.0899 & -0.0899 & -0.3141 \end{bmatrix}, \; B^* = \begin{bmatrix} 0 \\ 0.1450 \\ 0 \\ 0.2637 \end{bmatrix} \tag{5.124}
$$

Substitution of the average pi values gives a numerical result for the non-dimensional transfer function:

$$
\frac{\bar{y}(s^*)^*}{\delta_f(s^*)} = \frac{1}{s^{*2}} \frac{0.1450 \cdot s^{*2} + 0.0513 \cdot s^* + 0.0914}{s^{*2} + 0.6122 \cdot s^* + 0.1812} \tag{5.125}
$$

These normalized representations serve as basis system representations for the following controller design.

5.5.3 The Perturbation Model and Affine Matrix Representation

One measure of plant robustness is to determine how large a parameter variation the plant can tolerate. Under multiple, simultaneous parameter variations, it is important to choose variation combinations representative of actual system uncertainty. A first approach to classify the system uncertainty might be to define an uncertainty bound that spans all possible numerical permutations of the five pi-parameters, thus creating a hyper-cube in five-dimensional parameter space. However, this approach ignores key model information: it is difficult to have a light car with high moment of inertia, or a long car that has a small mass. A vehicle controller that allows such arbitrary parameter combinations ignores key parameter design constraints that underlie vehicle dynamics.

Investigation has shown that detailed measurements of vehicle parameters are unnecessary to obtain approximate estimates of vehicle behavior. In particular, examination of pi-plots (for instance, the π_3 versus π_4 plot earlier in this chapter) shows that the relationship between π_3 and π_4 falls almost along a line, and the other pi versus pi plots (not shown) show similar results. The five pi parameters span the five-dimensional parameter space almost *solely* along one dimension, with some scatter about this line. Thus, only *one* pi parameter needs to be measured to approximately predict the others.

Residual analysis (shown below) revealed that the scatter about the line follows a normal distribution. Methods were sought to numerically describe the line as well as the scatter about the line. The system dynamics are described in terms of the non-dimensional system matrices given above and in the Appendix of the thesis. These state-space matrices are composed of nonlinear pi-parameter functions, but since the base parameters are linearly interdependent and normally distributed, perhaps the nonlinear functions are similarly described. Because the 1-D line through parameter space contains a normal-distribution scatter about the line, additional perturbations must be added to each line so that all experimental data are included.

The π_3 parameter is chosen as the independent variable because this variable has the most variation due to velocity and cornering stiffness changes. Note that π_4 contains the same variations, but it was shown previously that π_4 is approximately represented as a linear function of π_3. Each pi-function in the non-dimensional matrices is written as a linear function of π_3 and some perturbation, with the linear form represented as a line with slope *m* and intercept *b*, and the perturbation sized to span the error. The state space system matrices can be rewritten as:

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\Pi_3 - f_1 & \Pi_3 + f_1 & f_2 \\ 0 & 0 & 0 & 1 \\ 0 & f_4 - f_3 & f_3 - f_4 & f_5 \end{bmatrix}, \mathbf{B}^* = \begin{bmatrix} 0 & \Pi_3 & 0 & f_3 \end{bmatrix}^T \qquad (5.126)$$

where

$$\begin{aligned} &\pi_3 \in [0, \infty] \\ &f_1 = \pi_4 \\ &\quad \approx m_1 \cdot \pi_3 + b_1 + \Delta_1 \\ &f_2 = \pi_2 \cdot \pi_4 - \pi_1 \cdot \pi_3 \\ &\quad \approx m_2 \cdot \pi_3 + b_2 + \Delta_2 \\ &f_3 = \pi_1 \cdot \pi_3 / \pi_5 \\ &\quad \approx m_3 \cdot \pi_3 + b_3 + \Delta_3 \\ &f_4 = \pi_2 \cdot \pi_4 / \pi_5 \\ &\quad \approx m_4 \cdot \pi_3 + b_4 + \Delta_4 \\ &f_5 = -\left(\pi_1^2 \cdot \pi_3 + \pi_2^2 \cdot \pi_4\right) / \pi_5 \\ &\quad \approx m_5 \cdot \pi_3 + b_5 + \Delta_5 \end{aligned} \qquad (5.127)$$

The set of compiled vehicle parameters (at the time) were used to generate 44 outputs for each of the five functions above; the best-fit line through these outputs was obtained using via simple regression. The table below summarizes the slope and intercept for the regression fit of each of the five functions of Equation 5.1.

If the residuals of the regression fit show a normal distribution about zero, then we can conclude that the span of vehicle systems across parameter space follows a normal distribution scatter about a 1-D line. As an example, the linear regression fit of function f_5 is shown in the figure below. A normal probability plot of the residuals is used to check for normality of the residuals. Again, as an example, the residuals of the f_5 function are shown. Clear normality

Table 5.10: Regression coefficients for the fit of nonlinear pi functions

f_I	Slope m_i	Intercept b_I	Δ_I Min	Δ_I Max
Π_3	1.000	0.000	N/A	N/A
1	0.818	0.130	-0.198	0.209
2	0.069	0.048	-0.155	0.156
3	2.005	-0.091	-0.311	0.365
4	2.398	0.059	-0.726	0.891
5	-2.262	0.019	-0.574	0.453

trends are demonstrated by the fact that the residuals are fit by a line connecting the 25 and 75 percentiles, and zero bias is demonstrated by the fact the line passes through the point (0,50%).

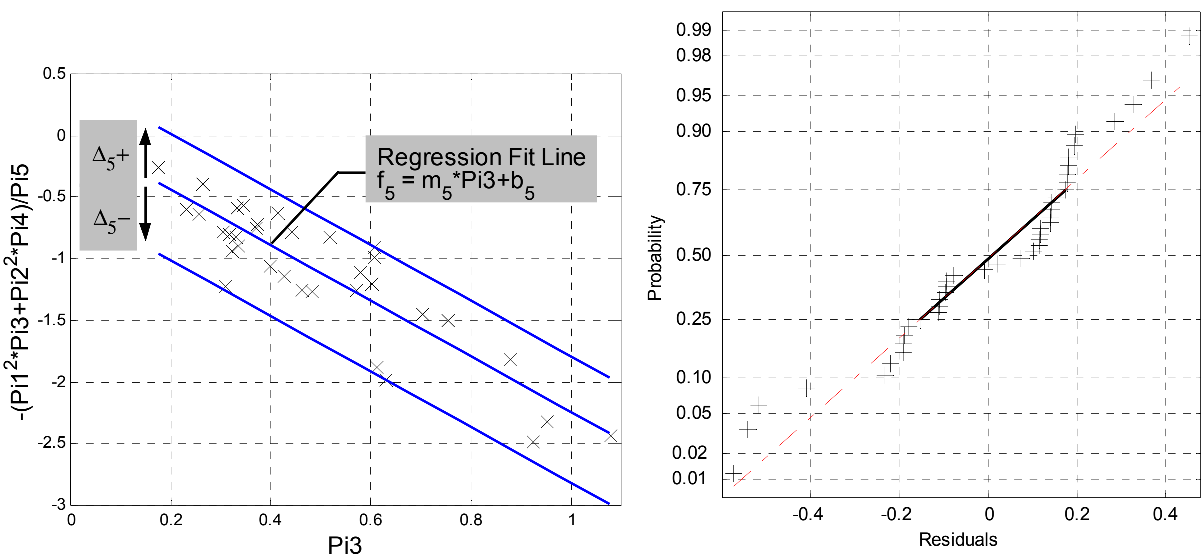


Figure 5.13: Regression fit and normal probability plot of linear approximation to nonlinear pi function

Although the perturbations still span 5 dimensions, the volume enclosed by the perturbations is now much smaller. For instance, the term π_4 ranges from 0.2 to 1.1, over a range of 0.9. However, the perturbation range needed for the linearized form of function f_1 (π_4) in Equations 5.1 reveals that only a 0.4 range is actually needed once the interdependence of π_3 and π_4 is accounted for.

Note that the system matrices **A*** and **B*** from the dimensionless state equations now have a linear (affine) dependence on π_3 and additional perturbation terms. The other four pi parameters and their variations have been absorbed into functions of π_3 and the five introduced perturbations. All system matrices can be written in the form of a matrix polynomial linearly dependent on π_3 and the perturbation terms. For instance:

$$\mathbf{A}^*(\pi_3, \Delta_1, \Delta_2, ..., \Delta_5) = \mathbf{A}^*_0 + \mathbf{A}^*_{\pi_3} \cdot \pi_3 + \mathbf{A}^*_1 \cdot \Delta_1 + \mathbf{A}^*_2 \cdot \Delta_2 + ... + \mathbf{A}^*_5 \cdot \Delta_5 \qquad (5.128)$$

The matrix **B*** can be rewritten in a similar manner. By substitution of each slope and intercept from Table 1 into the dimensionless matrices of the dimensionless system equation, the linearized, numerical values of each $\mathbf{A}^*_i$ in Equation (5.3) can be found to form dimensionless **A*** and **B*** matrices with linear parameter dependence on π_3 and perturbations.

The form of Equation (5.3) is called a Linear Differential Inclusion, and can be thought of in a more general form as a linear, parameter-dependent state-space representation of a vehicle system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{p}) \cdot \mathbf{x} + \mathbf{B}(\mathbf{p}) \cdot \mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \qquad (5.129)$$

where the **A***(***p***)* and **B***(***p***)* create linear combinations of system matrices, **S(p)**:

$$\begin{aligned} \mathbf{S}(\mathbf{p}) &= \begin{bmatrix} \mathbf{A}(\mathbf{p}) & \mathbf{B}(\mathbf{p}) \\ \mathbf{C} & 0 \end{bmatrix}, \\ \mathbf{S}(\mathbf{p}) &\in Co\{\mathbf{S}_1, ..., \mathbf{S}_n\} = \left\{ \sum_{i=1}^{n} \alpha_i \cdot \mathbf{S}_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\} \end{aligned} \qquad (5.130)$$

Each $\mathbf{S}_i$ is called a vertex system, and represents a vertex of the polytope envelope in matrix space. The term '*Co'* is used to denote that each system $\mathbf{S}_i$ is an element of a single convex hull. Therefore any system within the polytope envelope can be written in the form:

$$\mathbf{S}(\mathbf{p}) = \mathbf{S}_0 + \mathbf{p}_1 \cdot \mathbf{S}_1 + \mathbf{p}_2 \cdot \mathbf{S}_2 + ... + \mathbf{p}_n \cdot \mathbf{S}_n \qquad (5.131)$$

The vehicle system description under discussion is in fact described by such a system polytope. Note also that each $\mathbf{S}_i$ in Equation (5.6) does not necessarily need to represent a meaningful dynamic system; only their combination is relevant.

5.5.4 Performance Specifications

The performance specifications were to place the closed-loop poles in a particular pole region. To obtain valid non-dimensional pole-placement regions, it should be noted that a non-dimensional system will always have the same phase angle as the dimensional system. This is because the mapping of the pole locations from one domain to another requires only a temporal modification, which corresponds only to changing the scale of the s-plane axes. To define the closed loop responses via pole placement, we first impose damping constraints on the system: i.e. the non-dimensional poles were made to lie in a cone in the left-half s-plane with vertex on the origin and a vertex angle of $tan^{-1}(3\pi/4)$ rad. or ~67 degrees. This corresponds to a minimum damping ratio of 0.39, and a maximum step response overshoot of 26.4%.

In addition to the damping requirement, the closed-loop poles were sector bounded to lie in the region $-7 < s^* < -1$, in order to impose some lower and upper limit on the rise-time and system bandwidth. The relationship between the normalized s-space and traditional pole locations allows the dynamics of large cars to remain slow, and small cars to be fast. A time normalization of $s^* = (L/U) \cdot s$ is imposed; thus the pole constraint $s^* = -2$ has different meanings for different vehicles. A long car will inherently have slower dynamics than a small car, and the *s*-domain normalization causes a large car with slow dynamics to have the same s^* pole locations as a short car with fast dynamics, assuming both are traveling at the same speed. Thus, a single design constraint remains meaningful regardless of the particular vehicle size.

The pole placement region was numerically chosen with care: the lower limit on the pole regions is imposed to achieve a minimum rise time, while the upper limit is to prevent high-frequency dynamics and possible actuator saturation. The region itself represents an approximate bound on the *stable* poles of the 44 open-loop vehicle systems from published data. It is known from previous experience with vehicle control that a first choice of 'good' closed-loop pole locations is to choose regions near the stable open-loop pole locations. The pole locations of the non-dimensional and dimensional open-loop system corresponding to the vertices of the open-loop system matrix polytope are shown in the figure below.

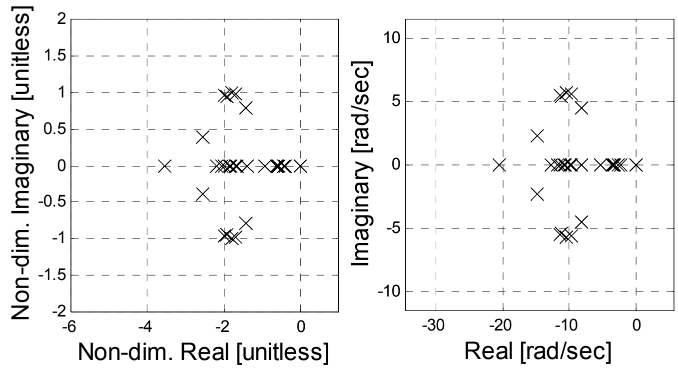


Figure 5.14: The open-loop pole locations of the vertex systems

Pole-placement design constraints can be imposed on polytopic systems by expressing the pole-placement region as a domain in the complex plane. The domain is characterized by:

$$D = \left\{ z \in C : \mathbf{L} + \mathbf{M} \cdot z + \mathbf{M} \cdot \bar{z} < 0 \right\} \tag{5.132}$$

where D is the complex pole region, C is the Complex s-plane, and $\mathbf{L} = \mathbf{L}^T$ and $\mathbf{M}$ are fixed real matrices. Relying on the results of Chilali and Gahinet (Chilali and Gahinet, 1996), the pole-placement objective can be re-cast as an existence argument for a common quasi-Lyapunov function. To do this, the matrices $\mathbf{L}$ and $\mathbf{M}$ are first represented by their elements, l_{ij} and m_{ij}, where the subscript notation is of the form *row,column*. The matrix $\mathbf{A}$ has all its eigenvalues in D if and only if there exists a positive definite matrix $\mathbf{P}$ such that:

$$\begin{gathered} \left[l_{ij}\mathbf{P} + m_{ij}\mathbf{AP} + m_{ji}\mathbf{PA}^T \right] < 0 \\ \text{for } i \geq 1, j \leq r \end{gathered} \tag{5.133}$$

with r equal to the number of rows of $\mathbf{A}$, and assuming $\mathbf{A}$ is square.

The above pole-placement result is extended to the polytopic system controller design problem by expressing the closed-loop system in autonomous form, i.e. a closed-loop system with a state-feedback controller expressed in the dynamics. The goal therefore is to find a $\mathbf{P}$ matrix and $\mathbf{K}$ gain vector satisfying:

$$\begin{gathered} \left[l_{ij}\mathbf{P} + m_{ij}\left(\mathbf{A}_p + \mathbf{B}_p \cdot \mathbf{K}\right)\mathbf{P} + m_{ji}\mathbf{P}\left(\mathbf{A}_p + \mathbf{B}_p \cdot \mathbf{K}\right)^T \right] < 0 \\ \mathbf{P} > 0 \\ \text{for } i \geq 1, j \leq r \end{gathered} \tag{5.134}$$

With $\mathbf{A}_p$ and $\mathbf{B}_p$ representing the system matrices of the polytope of system matrices expressed in Equation (5.4). Note that the above result is based on the formation of a common Lyapunov function; if a common Lyapunov function can be found for a set of state-space representations, then all share the same stability properties individually.

Finally, we note that the above equations are not linear in the decision variables **P** and **K**. A transformation, **Y** = **KP** linearizes the equations to create a linear matrix inequality. An LMI solver was then used to search over **Y** and **P** matrices to satisfy:

$$\begin{aligned}&\left[l_{ij}\mathbf{P}+m_{ij}\left(\mathbf{A}_p\mathbf{P}+\mathbf{B}_p\mathbf{Y}\right)+m_{ji}\left(\mathbf{P}\mathbf{A}_p{}^T+\mathbf{Y}^T\mathbf{B}_p{}^T\right)\right]<0\\&\mathbf{P}>0\\&for\ i\geq 1,\, j\leq r\end{aligned} \tag{5.135}$$

and the gain matrix, **K** was backed out by noting that $\mathbf{K}=\mathbf{Y}(\mathbf{X})^{-1}$. The MATLAB LMI toolbox contains several software tools to assist with the above design steps.

5.5.5 Implementation Results

When π_3 was allowed to range over the entire operating range of the vehicle, no solution to this problem could be found using the LMI tools just described. For the sake of computational feasibility, a stabilizing controller was sought that stabilized the given class of vehicles about a *particular* π_3 value in the presence of π_3 perturbations. As an example, the perturbations can be represented by a fixed π_3 value of 0.63 with π_3 perturbations allowable up to 0.03, and all other perturbations ranging over the values in Table 3. This controller should therefore stabilize any vehicle with π_3 values between 0.6 and 0.65 that is described by the vehicle distributions of Chapter 3, Section 3. This particular π_3 value corresponds to an average, full-size vehicle driving at a speed of 15 m/s (35 mph), and a corresponding scale vehicle speed of 1.95 m/s (See appendix). Note that the speed choice is somewhat arbitrary; changing the speed solely changes the nominal value of π_3. For this particular chosen operating speed, the non-dimensional gain matrix was found to be:

$$\mathbf{K}^*=\begin{bmatrix}8.1908 & 6.3391 & 7.7336 & 0.5499\end{bmatrix} \tag{5.136}$$

Using the vertex points of the perturbation hyper-cube, the closed-loop pole locations were determined in both dimensional and non-dimensional pole space. The resulting pole locations are shown in Figure 5.15.

The associated time-domain simulation plots of lateral position and yaw rate are shown in Figure 5.16 below for a series of closed loop vehicle lateral step responses. These responses correspond to a 1.3 m step-change in lateral position, and each step response corresponds to a vehicle representing one of the vertex systems.

To test the controller on an experimental vehicle, the Illinois Roadway Simulator (IRS) was utilized. For this scaled vehicle, a controller gain was obtained using the inverse dimensionless transformation presented in Chapter 3. The resulting gain, after substitution of the

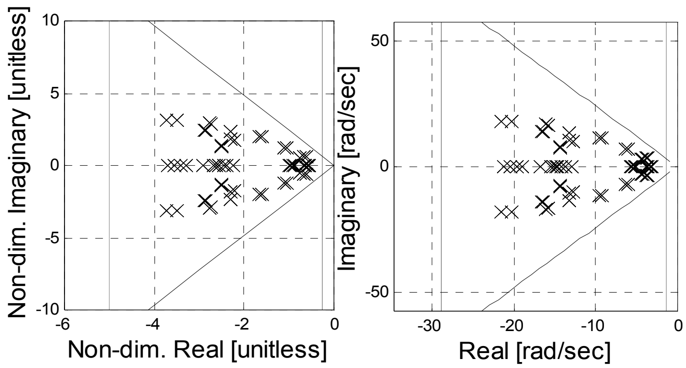


Figure 5.15: Closed-Loop pole locations for each vertex system

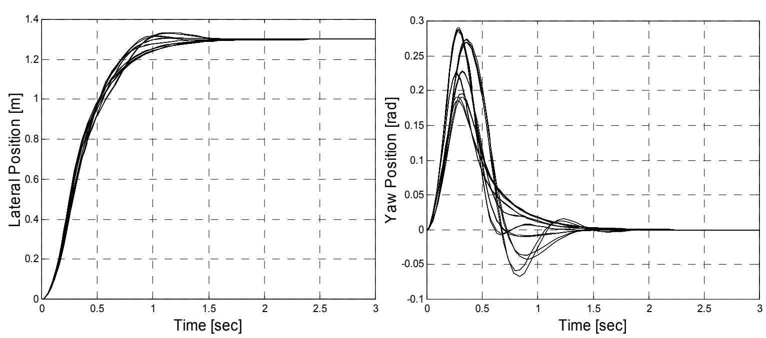


Figure 5.16: Step responses corresponding to each vertex system

vehicle parameter values with length L = 0.359 m and velocity 1.95 m/s, is shown below for the scale vehicle

$$\mathbf{K} = \begin{bmatrix} 22.85 & 3.2508 & 7.7336 & 0.1011 \end{bmatrix} \quad (5.137)$$

Experimental lateral position and yaw-rate responses were then obtained, as shown below. As can be seen in the figures, both the lateral position and yaw angle responses were within the bounds as predicted by the closed-loop pole locations and as predicted by the simulation step responses. It should be emphasized that **no vehicle system ID was required, other than confirmation that the vehicle conformed to the pi-distributions** presented in Chapter 3, Section 3.

The controller could be redesigned for different values of π_3 and different distributions. However, as mentioned earlier, under large variations in π_3 a solution to the optimization problem may not be guaranteed. Since π_3 varies primarily with cornering stiffness and vehicle

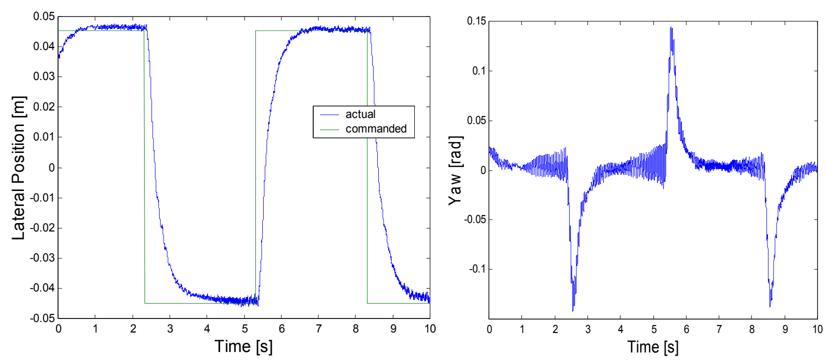


Figure 5.17: Experimental closed-loop system responses using IRS vehicle

speed, it may be possible to schedule robust controller designs with respect to vehicle velocity that can be readily measured. The robust controller would then be primarily required to incorporate and accommodate vehicle-to-vehicle structural variations in cornering stiffness. Thus, this controller approach may be quite useful as an initial controller methodology for stabilizing a new and unidentified vehicle.

5.5.6 Remarks on Robust Vehicle Control

The temporal and spatial re-parameterization of the linear vehicle Bicycle Model was shown to have several advantages over the traditional parameterization. First, the available model data have the appealing form of a Gaussian distribution about a line in the non-dimensional pi-space. This data suggests an 'average' and a 'standard deviation' of vehicle parameters. In addition, this allows vehicle-to-vehicle comparisons and numerically defines a parameter field over which a vehicle controller should be robust.

Second, a duality between velocity and cornering stiffness variation effects on vehicle dynamics has been suggested. The cornering stiffness variation can be cast as a road friction variation. Much work has been conducted in the vehicle controls community on each of these topics separately, and this work suggests perhaps these efforts can be unified in some manner.

Third, the dimensionless approach was used to discover physical relationships inherently present between vehicle parameters. As an example, the well known oversteer critical velocity was re-cast into the non-dimensional framework. It was demonstrated that simple functional forms such as lines could capture the physical relationships between non-dimensional parameters. In the case of vehicles, the pi-space is described fully by a multi-dimensional line with experimental data appearing to have a Gaussian distribution about this line.

Finally, the approach was used to obtain a robust controller where the perturbations were made with respect to non-dimensional parameters. Implementation of this controller was performed both in simulation and on scaled experiments. Within the specified variations of the non-dimensional parameters, closed loop performance characteristics can be specified.

5.6 Contributions of This Chapter

The primary contributions of this chapter are as follows, numbered by relation to corresponding sections of the chapter:

(1) The use of Dimensional Analysis simplified stability analysis of a system. Specifically,

   a. A nonlinear result of the Routh stability criteria for vehicle dynamics consisting of seven parameters was simplified to a linear criteria for stability consisting of two parameters.
   b. The above line representation correctly characterized the observed stability of actual vehicles; vehicles farthest from the above line were most stable, while vehicles closest to the above line were the least stable.

(2) The use of dimensionless parameters and dimensionless model representation made possible a complex model reduction problem of a heating and cooling systems. Namely,

a. Singular perturbation and other model-reduction techniques were analyzed from a dimensional standpoint. Based on matrix conditions, many (but not all) of the standard techniques were found to be valid only for dimensionless model representations.

b. The methods of dimensional analysis are found to be implicitly used by many authors, but in an incomplete manner that does not eliminate all sensitivity invariants and introduces new problem parameters.

(3) A model-order reduction of a analytically derived cooling system model is presented. The goal of the reduction was to preserve physical meaning, and a dimensionless representation specifically allowed comparison of systems with different state definitions.

(4) The use of dimensionless parameters is shown to *significantly* reduce the complexity of control problems that may be gain scheduled with respect to more than one parameter

a. A vehicle example is presented with the dual parameter variations of road friction and vehicle velocity are shown to be representable by a single gain-scheduled parameter

b. A generalized statement is made regarding the maximum number of parameters that can be coupled in such a dimensionless parameter approach. The number increases as the number of physical dimensions plus one.

c. An example is presented of a classical gantry system where a system of four parameters (two of them scheduling variables) is reduced to a dimensionless system of one parameter.

(5) To illustrate a primary advantage of dimensionless representations, a robust vehicle controller design is presented where model perturbations are modeled via dimensionless parameters.

a. A database of dimensionless parameters for vehicle dynamics was created from the literature.

b. The database revealed that the dimensionless system parameters are very interdependent; they tend to span the dimensionless pi-space almost exactly as a line (rather than a blob). This interdependence is conjectured to be due to common design constraints and a high level of design optimization.

c. A representation is chosen using perturbations about the nominal line through pi-space. The system dependence on the pi-parameters is approximated by an affine (linear) representation where the system matrices are linearly dependent on the model parameters.
d. The model perturbations were made wide enough to capture every vehicle in the database (approximately every vehicle in production)
e. To illustrate an ability to design controllers in a dimensionless framework, an LMI-based design found a solution to the control problem based on the affine representation, but also demonstrated that robust control over a wide variation in speeds (and road frictions) is not feasible.
f. The controller was transformed back into the physical domain via simple transforms.
g. An experimental implementation was presented demonstrating the controller on the vehicle testbed of Chapter 2.

5.7 References

1. Brennan, Sean . "Modeling and Control Issues Associated With Scaled Vehicles." Masters Thesis. University of Illinois at Urbana-Champaign, 1999.
2. Chilali, M., and P. Gahinet. "H-Inf Design With Pole Placement Constraints: An LMI Approach." IEEE Trans. Automat. Control 41 (1996): 358-67.
3. Frank, Paul M. Introduction to System Sensitivity Theory. New York: Academic Press, 1978.
4. Franklin, Gene F., J. David Powell, and Abbas Emami-Naeini. Feedback Control of Dynamic Systems. 1986. Fourth ed. Upper Saddle River, New Jersey: Prentice Hall, 2002.
5. Guldner, J., H. S. Tan, and S. Patwardhan. "Analysis of Automatic Steering Control for Highway Vehicles With Look-Down Lateral Reference Systems." Vehicle System Dynamics 26.4 (1996): 243-69.
6. Hart, George W. Multidimensional Analysis, Algebras and Systems for Science and Engineering. New York: Springer-Verlag, 1995.

7. Horiuchi, Shinichiro, Naohiro Yuhara, and Akihiko Takei. "Two Degree of Freedom H-Infinity Controller Synthesis for Active Four Wheel Steering Vehicles." Vehicle System Dynamics Supplement 25 (1996): 275-92.

8. Khalil, Hassan K. Nonlinear Systems. Second ed. Upper Saddle River, NJ 07458: Prentice Hall, 1996.

9. Kokotovic, Petar V. Singular Perturbation Methods in Control: Analysis and Design. Orlando, Florida: Academic Press, 1986.

10. Naidu, D. S. Singular Perturbation Methodology in Control Systems. London, United Kingdom: Peter Peregrinus, 1988.

11. Ono, Eiichi, et al. "Vehicle Integrated Control for Steering and Traction Systems by Mu-Synthesis." Automatica 30.11 (1994): 1639-47.

12. Rassmussen, Bryan Philip. "Control-Oriented Modeling of Transcritical Vapor Compression Systems." Masters Thesis. Univerisity of Illinois at Urbana-Champaign, 2002.

13. Sastry, Shankar. Nonlinear Systems: Analysis, Stability, and Control. New York: Springer-Verlag, 1999.

14. Shiotsuka, T., A. Nagamatsu, and K. Yoshida. "Adaptive Control of 4WS System by Using Neural Network." Vehicle System Dynamics (1993): 411-24.

15. Shladover, S. " Review of the State of Development of Advanced Vehicle Control Systems (AVCS)." Vehicle System Dynamics 24(6-7) (1995): 551-95.

16. Skogestad, Sigurd, and Ian Postlethwaite. Multivariable Feedback Control, Analysis and Design. 1996. Baffins Lane, Chichester England: John Wiley & Sons Ltd., 2000.

17. Tagawa, Y., et al. "Robust Active Steering System Taking Account of Nonlinear Dynamics." Vehicle System Dynamics Supplement 25 (1996): 668-81.

18. Wong, J. Y. Theory of Ground Vehicles. 2 ed. New York: J. Wiley & Sons, 1993.

19. Zhou, K., J. Doyle, and K. Glover. Robust and Optimal Control. Upper Saddle River, NJ: Prentice Hall, 1996.

Chapter 6

A Linear Dynamics View of Dimensional Analysis and Control

The generation of a dimensionless equation written only in terms of pi-parameters is a process known as *normalization* in the field of dimensional analysis (for instance, see Kline's book on the topic, (Kline, 1965) (p.70). Unfortunately, the use of the word normalization in the controls context implies a very specific and different meaning. As the usage of the word in dimensional analysis predates the controls usage by nearly a century, the word normalization hereafter is meant to imply *creation of a dimensionless governing equation using relevant dimensional measures*. In other fields such as Physics, the dimensional meaning (versus mathematical meaning) is also the accepted usage. For instance, the renormalization of the Standard Model at scales lower than the quark length impliefs that a new measure of space and time will be used at these scales.

6.1 History of Differential Equation Approaches to Dimensional Analysis

Many have used differential forms to determine normalized equations and governing parameters. The earliest formal treatment is attributed to Ruark (Ruark, 1935) by Kline (Kline, 1965), but an expanded version is presented by Birkhoff (Birkhoff, 1948). Kline states conditions for applicability of pi-analysis to mathematical equations, namely that the system of equations is well-posed, i.e. (1) a unique solution exists with the boundary conditions as given and (2) that the solution is continuously dependent on the boundary conditions. The first criterion follows from the requirement that the equation describe a physical system, which is assumed to exist only in one state at a time (therefore, we may relax this condition somewhat). The need for the second criterion was demonstrated in Chapter 4 on sensitivity analysis. Since Kline's work, Taylor has presented a generalized method dimensional analysis on homogenous differential equations that is presented shortly.

In general, the following rules of normalization are suggested by Kline (Kline, 1965) (p. 71):

1. Attempt to define a dependent and dimensionless variable so that it is approximately unity over a finite distance and nowhere exceed unity in the domain of concern.
2. Attempt to define all independent dimensionless variables so that their increment is approximately unity over the same domain of concern.

In general, the problem boundary conditions and domain of the problem (temporal, mass, or spatial) are good starting points for choosing normalization dimensions. Note that the criteria of unit increment and unit output correlate very strongly with empirical techniques already 'discovered' by others in dealing with Robust Control problems, for instance see Sgoestad and Postlethwaite (Skogestad and Postlethwaite, 2000) (Chapter 1).

Example: Temperature Distribution in a 3-D Solid

As an example, we consider a thermal problem presented in Kline. The problem is to determine the temperature profile inside a 3-D homogenous object, where the heat transfer is primarily by conduction. The governing equation is given by Fourier's relation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (6.1)$$

where T is the temperature, x, y, z are the Cartesian coordinates, t is the time, and α is the thermal diffusivity. The problem is subject to boundary conditions that the initial temperature is T_i and the temperature at surfaces $x=0, x=L, y=0, y=M, z=0, z=N$ are T_a for positive times. We normalize the system equation by choosing new coordinates

$$\overline{T}=\frac{T}{T_i}, \overline{x}=\frac{x}{L}, \overline{y}=\frac{y}{M}, \overline{z}=\frac{z}{N} \qquad (6.2)$$

note that we measure all temperatures from T_a as a datum. For time, we choose some characteristic time, t_c, that it takes the particles at x, y, z to achieve some fraction of their final temperature (say 2/3): $\overline{t} = \frac{t}{t_c}$. As noted by Kline, the time normalization can be chosen in terms of some more meaningful constant such as the time constant. This, however, requires a solution (which may not be available) and the resulting equations are easily mapped between different scale factors, and no new results will be obtained. The normalized equation is:

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \left(\frac{L}{M}\right)^2 \cdot \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} + \left(\frac{L}{N}\right)^2 \cdot \frac{\partial^2 \overline{T}}{\partial \overline{z}^2} = \frac{L^2}{t_c \cdot \alpha} \cdot \frac{\partial \overline{T}}{\partial \overline{t}} \qquad (6.3)$$

The pi-values governing the equation can be read directly as coefficients to the equation. Note also that the dimensionless parameters do not necessarily have to remain fixed. Changing these values corresponds to moving through the domain of interest for the particular problem or system.

If a group of systems obeys the same normalized governing equations and boundary conditions, then the group is called (by the author) a *class of systems* or *dimensional class*. This is again consistent with the definition presented earlier and in usage in the field.

It must be emphasized that no references have been found dealing with the generalized, control-based dimensional analysis of normalized equations, linear or nonlinear, of a system dynamic equation. All of the previous citations deal solely with autonomous systems, and the inclusion of control inputs require additional consideration which are presented in this work.

6.2 Differential Equation Generalizations Related to Control

6.2.1 Homogenous Equations

In the most general form, a homogenous linear differential equation is given by the following form (Taylor, 1974)(p. 95):

$$a_n \cdot \frac{d^n x}{dt^n} + \ldots + a_{n-k} \cdot \frac{d^{n-k} x}{dt^{n-k}} = 0 \qquad (6.4)$$

One can generate a set of dimensionally constraining equations by noting that the quotient of any pair of coefficients has the dimensions of the independent variable raised to the power equal to the difference in the order of the terms, thus:

$$\frac{a_n}{a_{n-k}} \mp y^k \qquad (6.5)$$

One may form a dimensionless parameter given any three coefficients, i.e.

$$\left(\frac{a_n}{a_{n-k}}\right)^m \cdot \left(\frac{a_{n-m}}{a_n}\right)^k \mp 1 \qquad (6.6)$$

Therefore, we may conclude that the number of independent dimensionless groups which can be formed from the coefficients of a homogenous linear differential equation is equal to the number of terms minus two (Taylor, 1974).

6.2.2 Similarity of Newtonian Systems and Lagrangian Dynamics

For two Newtonian systems, geometric similarity of two paths require a correspondence between measurements of distance from the origins of the two system as:

$$\overline{O_2P_2} = \lambda \cdot \overline{O_1P_1} \qquad (6.7)$$

or a direct length scaling. If we next require that two points in the different systems share the same direction of body forces with magnitude scaled by a factor, we obtain a requirement:

$$f_2 = \mu \cdot f_1 \qquad (6.8)$$

were μ is a force scaling factor and the two forces are assumed to be measured at corresponding points. If we now suppose that, as two particles traverse similar paths, they do so in ratios of velocities given by:

$$v_2 = \nu \cdot v_1 \qquad (6.9)$$

then we now have sufficient constraints to establish dimensional scaling criteria on all remaining measured quantities. For instance, the above assumptions constrain the ratio of times to travel a given path segment to be:

$$\frac{t_2}{t_1} = \frac{\lambda}{\nu} \tag{6.10}$$

The velocity ratios may also be given by a ratio of times multiplied by the ratio of accelerations:

$$\frac{\lambda \cdot \mu}{\nu} = \nu \Rightarrow \frac{\lambda \cdot \mu}{\nu^2} = 1 \tag{6.11}$$

Thus, the assumed velocity, distance, and force ratios must satisfy this constraint in order to maintain similarity of velocities, distances, and forces

We now consider Newton's Law of Forces to show that the study of similarity of systems governed by this law yield the same requirements (Duncan, 1953) (p. 29). Beginning with the standard form of the third law, we may write:

$$\frac{d^2x}{dt^2} = \sum F_x \tag{6.12}$$

where force components are delineated along a particular dimensional axis. Let $\frac{dx}{dt} = u$, so that the dimensions become more obvious:

$$\frac{d^2x}{dt^2} = \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = u \cdot \frac{du}{dx} = \sum F_x \tag{6.13}$$

assuming that the force on a moving mass is a function of the trajectory, we may write for two different systems of unit mass:

$$\begin{aligned} u_1 \cdot \frac{du_1}{dx_1} &= f_{x_1}(x_1) \\ u_2 \cdot \frac{du_2}{dx_2} &= f_{x_2}(x_2) \end{aligned} \tag{6.14}$$

From the assumptions of 6.7 and 6.8, we may write:

$$\begin{aligned} x_1 &= \frac{x_2}{\lambda} \\ f_{x_2}(x_2) &= \mu \cdot f_{x_1}(x_1) \end{aligned} \tag{6.15}$$

giving:

$$u_2 \cdot \frac{du_2}{dx_2} = \mu \cdot f_{x_1}\left(\frac{x_2}{\lambda}\right) \tag{6.16}$$

Under assumptions of 6.9, we then require:

$$\left(\frac{v^2}{\lambda \cdot \mu}\right) u_1 \cdot \frac{du_1}{dx_1} = f_{x_1}\left(x_1\right) \tag{6.17}$$

This gives the same similarity constraint as derived earlier, namely matching of the Froude number.

Other authors, notably Duncan (p. 59), have noted that the one can also use the formulation of Lagrangian dynamics in the dimensionless form to directly obtain requirements for physical similarity of Newtonian systems derived above. Specifically, the typical Lagrangian equation applied to one axis generally gives:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = Q_r \tag{6.18}$$

Here, T and V are the kinetic and potential energies (respectively) and Q_r is the generalized force corresponding to direction q_r. We form a dimensionless form by defining dimensionless forms of time, kinetic energy, potential energy, and force:

$$\tau = \frac{U \cdot t}{L} \qquad \bar{T} = \frac{T}{\rho \cdot U^2 L^3}$$
$$\bar{V} = \frac{V}{g \cdot \rho \cdot L^4} \qquad \bar{Q}_r = \frac{Q_r}{g \cdot \rho \cdot L^4} \tag{6.19}$$

Here U is a constant velocity associated with the system or some constant having units of velocity, L is a typical linear dimension of the system, ρ is a constant density of the system, and g is a constant body force. The dimensionless form of the equation becomes:

$$\left(\frac{U^2}{L \cdot g}\right)\left[\frac{d}{d\tau}\left(\frac{\partial \bar{T}}{\partial \dot{q}_r}\right) - \frac{\partial \bar{T}}{\partial q_r}\right] + \frac{\partial \bar{V}}{\partial q_r} = \bar{Q}_r \tag{6.20}$$

We therefore conclude that similarity of a Froude number is a requirement for physical similarity of Newtonian systems, a fact confirmed earlier for the specific case of matching vehicle dynamics.

6.3 Controller Normalizations

To demonstrate that controller design using the dimensionless coordinates is straightforward, consider classical pole-placement on a full-sized vehicle. The following published (Alleyne, 1997) vehicle parameters were used: m = 1670 kg, I_z = 2100 kg-m^2, a = 0.99 m, b = 1.7 m, $C_{\alpha f}$ = 123,190 N/rad, $C_{\alpha r}$ = 104,190 N/rad, and U = 15 m/s. The goal of this simple lateral positioning controller is to place the closed-loop poles at $K = [-10, -15, -20, -25]$ (rad/sec) in the s-space, corresponding to $K = [-1.79, -2.69, -3.59, -4.48]$ (unitless) in normalized s-space. The gain obtained from performing non-dimensional pole placement is K*, defined by the following relationship:

$$u^* = K^* \cdot x^* = u = K \cdot x = K(M \cdot x^*) \Rightarrow K^* = K \cdot M \qquad (6.21)$$

The gain matrix obtained using traditional controller pole-placement is: $K = [7.62, 0.712, 5.70, -0.0856]$. The gain obtained by performing non-dimensional pole-placement is: $K = [20.5, 10.68, 5.70, -0.478]$. Using the conversion in Equation 6.18, the non-dimensional gain-matrix predicts that the dimensional gain matrix should be $K = [7.62, 0.712, 5.70, -0.0856]$, exactly as predicted. It should therefore be clear that whether the controller design is conducted in non-dimensional space or in classical dimensional space, the resulting gains are equivalent as long as the gains account for appropriate dimensional and temporal conversions.

6.4 Robust Control by Dynamic Methods of Uncertainty Description

Previously in Chapter 5, a robust lateral-position controller was presented useful for highway driving of any passenger vehicle based on dimensional scaling. The system uncertainty was represented as matrix element perturbations of the system matrices, and a Linear Matrix

Inequality (LMI) approach was then used to design a state-feedback lateral position controller robust to expected parameter variation between vehicles. The resulting controller therefore robustly stabilizes all vehicles dynamically described by the Bicycle Model and which are parametrically bounded by fixed bounds. An important result of that study was the conclusion that a state-feedback controller is not capable of robust lateral vehicle positioning over wide variations in velocity without some type of gain scheduling. Unfortunately, the robustness bounds do not address dynamic uncertainty associated with unmodeled dynamics, disturbances, or measurement noise, and therefore a different approach is necessitated should one wish to account for these.

This work develops an alternative, dimensionless representation of vehicle dynamics that is more suitable for a generalized vehicle analysis by using a frequency-domain representation of model uncertainty. Again, the perturbations about the average are easily developed that reasonably encompass all production vehicles. These uncertainty bounds are then used to generate a robust controller suitable for any vehicle. For the purposes of demonstration, the focus of this work is a lateral-positioning control task. The resulting controller is again demonstrated on a scaled experimental vehicle.

6.4.1 System Model

A full-fidelity vehicle model is dependent upon a multitude of parameters, however here we focus on a set of 'primary' parameters that govern the chassis motion by utilizing the simplest model that captures the dynamics of interest. For this work, the model of interest is commonly referred to as the Bicycle Model (from Chapter 2) and is dependent on the following parameters:

Table 6.1: Test vehicle parameters

m = vehicle mass	(5.451 kg)
I_z = vehicle moment of inertia	(0.1615 kg·m^2)
U = vehicle longitudinal velocity	(2.95 m/s)
a = distance from C.G. to front axle	(0.1461 m)
b = distance from C.G. to rear axle	(0.2191 m)
L = vehicle length, = $a + b$	(0.3652 m)

$C_{\alpha f}$ = cornering stiffness of front 2 tires (65 N/rad)

$C_{\alpha r}$ = cornering stiffness of rear 2 tires (110 N/rad)

The values in parenthesis are quite different from a typical full-sized vehicle because they correspond to the measured values for a 1/7-scale experimental scale used on the Illinois Roadway Simulator which is a treadmill/vehicle counterpart to a wind tunnel/airplane testing system.

From the distributions of Chapter 3, the average pi parameters are obtained for full-sized production vehicles:

$$\bar{\pi}_1 = 0.4431, \bar{\pi}_3 = \frac{145.6771}{\mathrm{U}^2}, \bar{\pi}_4 = 1.0977 \cdot \bar{\pi}_3, \bar{\pi}_5 = 0.2510 \qquad (6.22)$$

These average values provide the dynamics of an average vehicle when substituted into the bicycle model, which is discussed shortly. The normalizations of Chapter 4, Section 3 are then applied to the standard vehicle-dynamic model to illustrate how the dimensional transformations and parameter distributions both serve to define an average vehicle dynamic. First, reconsider the standard state-space description of planar vehicle dynamics, which are commonly referred to as the Bicycle Model (again, see Chapter 2). The state vector is defined as [lateral position, lateral velocity, yaw angle, yaw rate], or:

$$\mathbf{x} \equiv \begin{bmatrix} y & \frac{dy}{dt} & \psi & \frac{d\psi}{dt} \end{bmatrix}^T \qquad (6.23)$$

and front steering input, $\mathbf{u} \equiv \left[\delta_f \right]^T$, as the sole control channel. Note that all states are measured with respect to the vehicle's center-of-gravity in what are known as 'body-fixed' coordinates. In this work, an error preview scheme is utilized, represented by a modification of the **C** matrix with the inclusion of a preview distance, as presented in Patwardhan, et.al (Patwardhan, Tan, and Guldner, 1997). The traditional state space form of the Bicycle Model in the form of Chapter 2 is modified to include a small preview:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{f_1}{mU} & \dfrac{f_1}{m} & \dfrac{-f_2}{mU} \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-f_2}{I_z \cdot U} & \dfrac{f_2}{I_z} & \dfrac{-f_3}{I_z \cdot U} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{C_{\alpha f}}{m} \\ 0 \\ \dfrac{a \cdot C_{\alpha f}}{I_z} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ d_s \\ 0 \end{bmatrix}^T, \mathbf{D} = [0] \qquad (6.24)$$

with:

$$f_1 = C_{\alpha f} + C_{\alpha r}, f_2 = a \cdot C_{\alpha f} - b \cdot C_{\alpha r}, f_3 = a^2 \cdot C_{\alpha f} + b^2 \cdot C_{\alpha r} \qquad (6.25)$$

In the linear description, the effect of the preview distance d_s is equivalent to adding yaw angle feedback, and therefore preview is simply a special case of state feedback (Guldner, Tan, and Patwardhan, 1996; Patwardhan, Tan, and Guldner, 1997; Peng and Tomizuka, 1993).

To pick the value of d_s, a wide range of preview values were examined. The performance effect of increasing the preview distance was primarily to decrease the significant phase margin deficiency caused by the two free integrators; for explanation, see (Guldner, Tan, and Patwardhan, 1996; Patwardhan, Tan, and Guldner, 1997; Peng and Tomizuka, 1993). The robustness effect of increasing preview is to increase high-frequency model uncertainty, an effect of secondary concern since the difficulty with vehicle robustness is a problem governed generally by low-frequency uncertainty. Rather than include preview as an additional unnecessary design varaible, a fixed preview distance of 2 vehicle lengths was selected and is used hereafter. This does not affect the central focus of the current work.

To obtain the time conversion for the dimensionless vehicle system, we choose a new time unit, $\mathbf{S}$, from Section 4.3 as:

$$\mathbf{S} = \frac{L}{U} \qquad (6.26)$$

and new measurement units for the states, $\mathbf{M}$, from Section 4.3 as:

$$\mathbf{M} = diag\begin{bmatrix} L & U & 1 & \dfrac{U}{L} \end{bmatrix} \qquad (6.27)$$

Using the dimensionless state-space transformation results of Chapter 3, the state-space matrices become:

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -p_1 & p_1 & -p_2 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-p_2}{\pi_5} & \frac{p_2}{\pi_5} & \frac{-p_3}{\pi_5} \end{bmatrix}, \mathbf{B}^* = \begin{bmatrix} 0 \\ \pi_3 \\ 0 \\ \frac{\pi_1 \cdot \pi_3}{\pi_5} \end{bmatrix}, \mathbf{D}^* = [0] \qquad (6.28)$$

$$p_1 = \pi_3 + \pi_4, p_2 = \pi_1\pi_3 - \pi_2\pi_4, p_3 = \pi_1^2\pi_3 + \pi_2^2\pi_4 \qquad (6.29)$$

Note that *all* terms in the system matrices are represented by pi values; this is always true in the process of nondimensionalization. In the nondimensional form, the **C*** matrix becomes:

$$\mathbf{C}^* = [1 \quad 0 \quad d_s / L \quad 0]. \qquad (6.30)$$

with the term:

$$\pi_6 = d_s / L . \qquad (6.31)$$

as a new pi variable. Note that $\bar{\pi}_6 = 2$ for purposes of our discussion. A similar dimensionless representation can be obtained in the Laplace domain, generating transfer function dynamics solely dependent on the vehicle pi parameters.

To complete the vehicle model description, it is useful to add additional scaling transforms to limit the largest control effort, tracking error, and reference input to all have unity infinity-norms. To do this, one uses a variable transformation suggested by Skogestad and Postlethwaite (Skogestad and Postlethwaite, 2000). For the vehicle system, reasonable signal norms are:

$$\begin{aligned} u_{max} &= \mathbf{D}_u = 0.1745[rad](=10[degrees]) \\ e_{max} &= \mathbf{D}_e = 0.15[m](=0.5[scale\ lanes]) \end{aligned} \qquad (6.32)$$

where a 'scale lane' is defined as the width of a driving lane as measured on the scale vehicle. The above limits become in the dimensionless parameter space:

$$\begin{aligned} u^*_{max} &= 0.1745[unitless] \\ e^*_{max} &= \frac{e_{max}}{L} = 0.4184[unitless] \end{aligned} \qquad (6.33)$$

With the signals normalized as above, the goal is to maintain an output position within [-1,1], using control inputs bounded by [-1,1], given a reference input that remains within [-1,1].

To specify the average dynamics from Equation 6.28, the pi values are then fixed to the average values as given in the vehicle distributions of Chapter 4 with $\pi_3 = 0.5$. This pi-term must be fixed as it cooresponds to the velocity scheduling parameter. The value of 0.5 corresponds to 3.0 m/s scale speed, and about 42 mph for a full-sized vehicle. The dimensionless transfer function for the nominal system is given by:

$$\frac{\overline{y}(s^*)^*}{\delta_f(s^*)} = \frac{1}{s^{*2}} \frac{2.2465 \cdot s^{*2} + 2.9609 \cdot s^* + 1.1563}{s^{*2} + 2.1923 \cdot s^* + 1.5797} \tag{6.34}$$

Note that $s \Rightarrow s^*$ because s has dimensions of $1/t$ and must also be normalized. With signal normalization as described in Chapter 3, the transfer function becomes:

$$\frac{\hat{\overline{y}}_n^*(s^*)}{\delta_f(s^*)} = \frac{\overline{y}^*(s^*)}{\delta_f(s^*)} \cdot \frac{u^*_{max}}{e^*_{max}} = \frac{1}{s^{*2}} \frac{0.9546 \cdot s^{*2} + 1.2582 \cdot s^* + 0.4913}{s^{*2} + 2.1923 \cdot s^* + 1.5797} \tag{6.35}$$

6.4.2 Robust Controller Design

For a single controller to be robust enough to be used on any passenger vehicle, numerical bounds are required on the model uncertainty caused by variations in pi parameters. A poor representation of plant uncertainty would be to examine perturbations in the average vehicle dynamic by performing one-at-a-time perturbations in the pi-parameters. As noted in earlier sections, true vehicle parameters are highly interdependent because vehicle designs are highly optimized toward common design criteria. As a result, the pi parameters only exist along a line passing through pi-space; robustness for parameter deviations far from this line would be unnecessarily cautious. In the following we use this to develop a better, data-driven approach with an appropriate but simple perturbation representation.

A better technique than one-at-a-time perturbations is to utilize the vehicle database (presented in the Appendix) directly to compare the relative error between the average vehicle and each individual database member. The frequency-dependent error, e(*jw*), between the average plant and the i^{th} plant is then given by:

$$e(jw) = \left| \frac{G\left(jw, \pi_{1i}, \ldots, \pi_{5i}\right) - G\left(jw, \bar{\pi}_1, \ldots, \bar{\pi}_5\right)}{G\left(jw, \bar{\pi}_1, \ldots, \bar{\pi}_5\right)} \right| \tag{6.36}$$

Where *G(jw)* represents the frequency response of the vehicle bicycle model dependent on the pi parameters. A simple multiplicative uncertainty description is used to describe this system variation, represented in block-diagram form in the figure below.

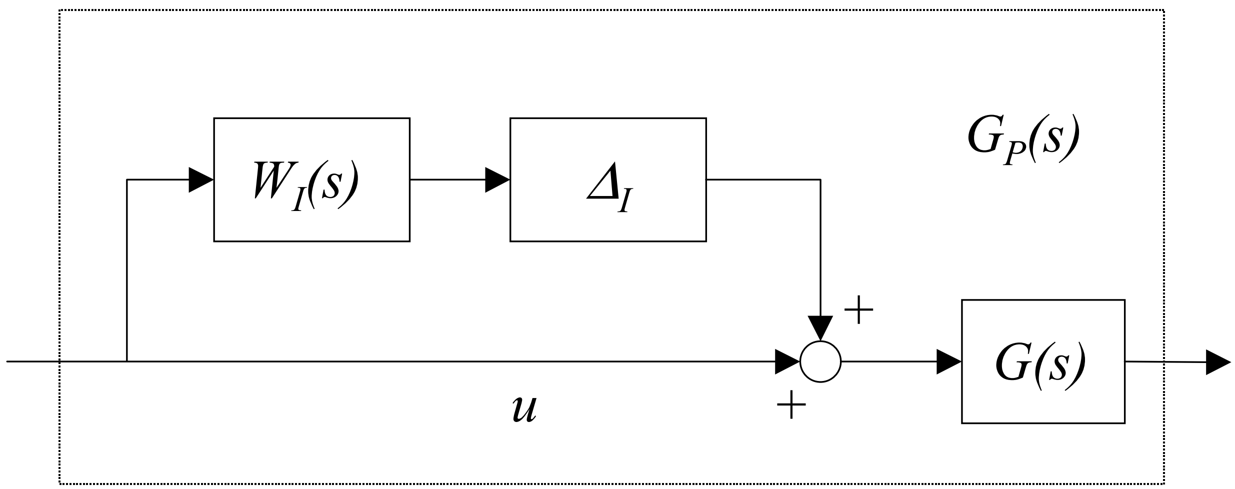


Figure 6.1: Multiplicative uncertainty model

The plot of each plant deviation calculated from 6.36 for each vehicle in the database is shown in Figure 6.2 with a fixed value of $\pi_3 = 0.5$. It is clear that the maximum multiplicative uncertainty is approximately constant in the low frequency region, a result that justifies a multiplicative representation.

If the multiplicative uncertainty is greater than unity at low frequencies, then by Figure 6.1 the plant may potentially have zero gain, and the robust control problem is not solvable. As we see in the uncertainty bounds of Figure 6.1, the robust vehicle control problem is not solvable if π_3 is much smaller than 0.2. The relationship between π_3 and velocity is show in Figure 6.4. The high and low velocities of depend on the lowest and highest reported cornering stiffness values for the vehicles in the database of the Appendix.

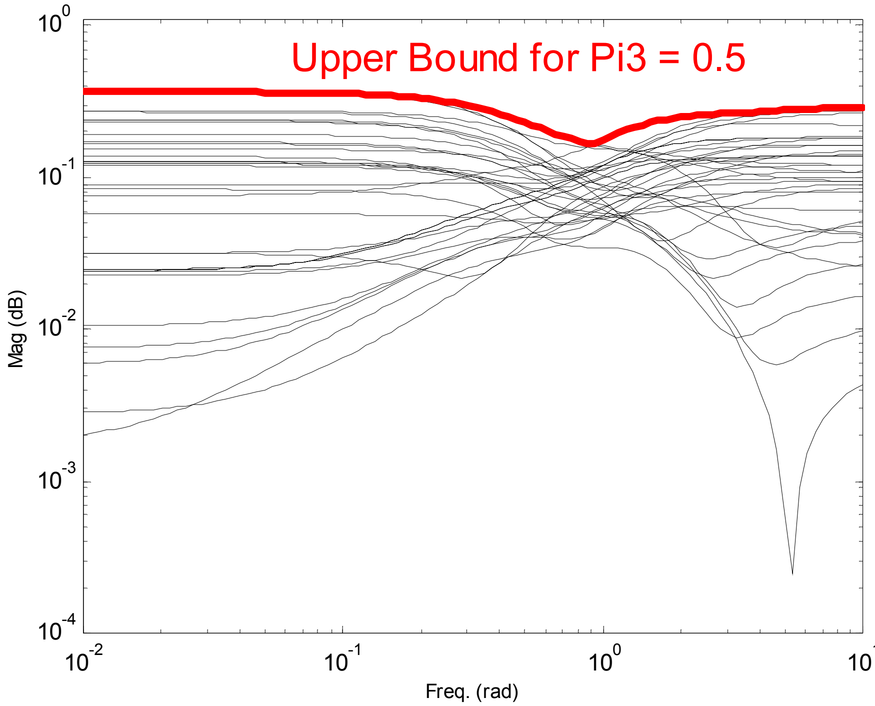


Figure 6.2: Multiplicative uncertainty bounds, $\pi_3 = 0.5$, showing all vehicles

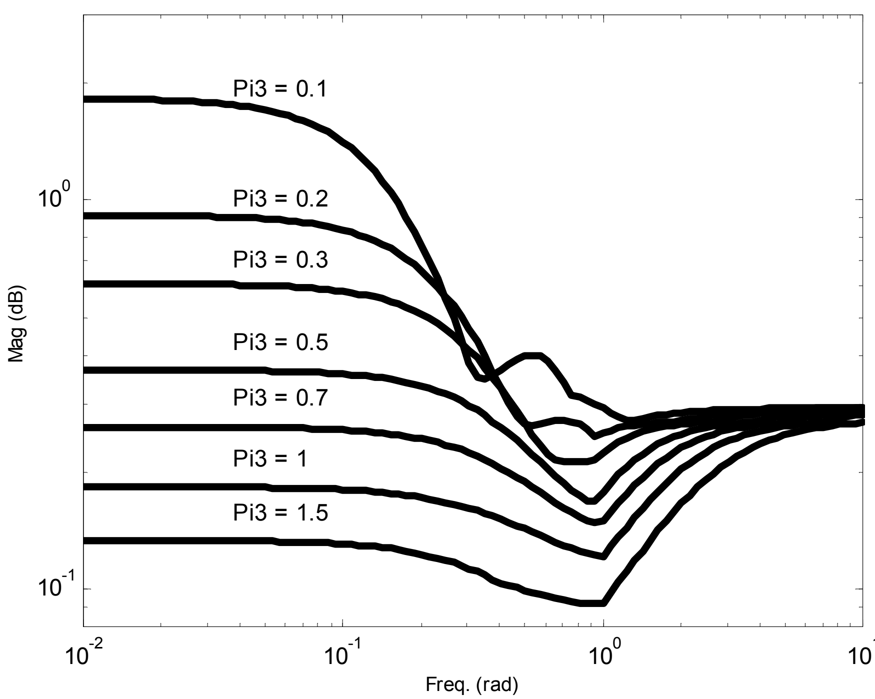


Figure 6.3: Multiplicative uncertainty bounds for various π_3 values

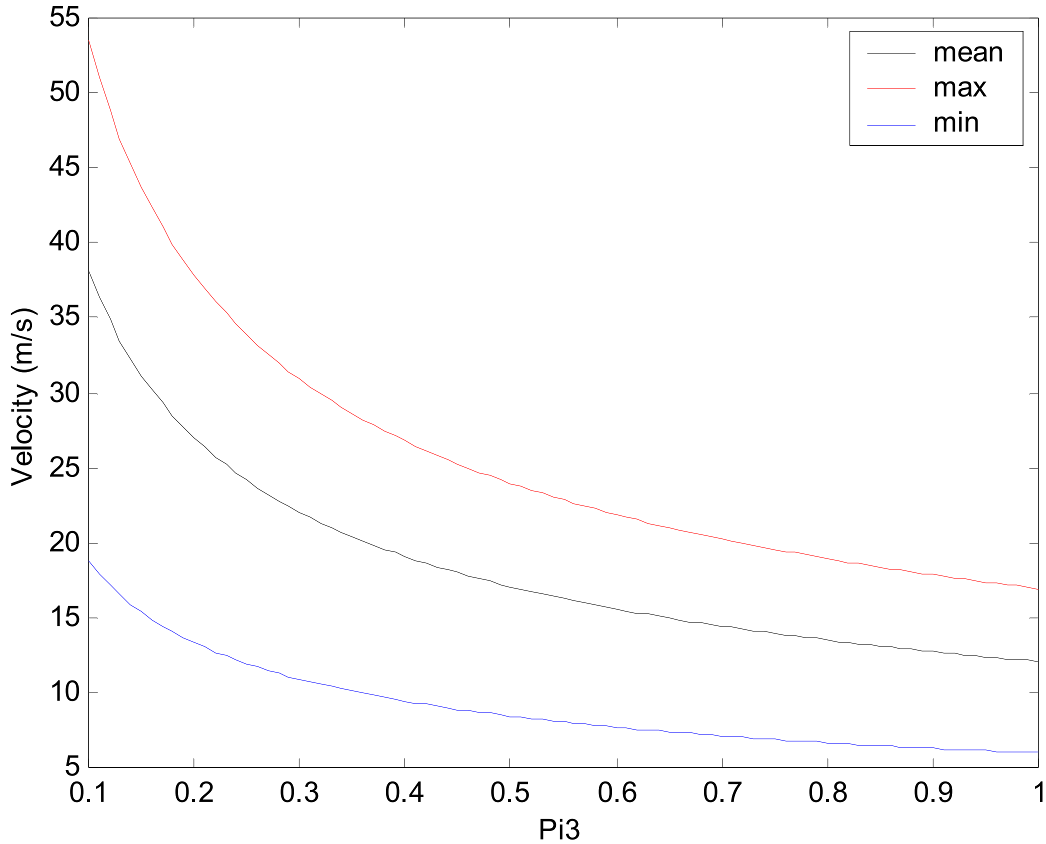


Figure 6.4: Relationship between pi3 and velocity for various velocities

For $\pi_3 = 0.2$, the corresponding velocity ranges from a max of 37.9 m/s (85 mph) to a minimum of 13.3 m/s (30 mph), with an average of about 27.0 m/s (60 mph). Thus, a vehicle-general controller is not feasible above an average speed of 27 m/s without more detailed representation of the model uncertainty.

For the remainder of the controller design, the π_3 parameter was fixed at 0.5, a value representing a full-sized vehicle on average road surface at 40 mph. At this speed, the vehicle-to-vehicle multiplicative uncertainty bound is less than unity at all frequencies, so a robust controller design should be feasible. However, uncertainty is experimentally known to increase at high frequencies due to unmodeled dynmaics. The weight representing system robustness, w_I, is chosen in anticipation of these dynamics. In this case:

$$w_I(s^*) = \frac{0.2 \cdot s^* + 0.5}{0.1 \cdot s^* + 1} \tag{6.37}$$

The high frequency gain was made slightly higher than needed in order to account specifically for unmodeled steering actuator dynamics and vehicle roll and pitch effects. Since the H-infinity system representation does not allow unstable open-loop systems, the double integrator in Equation 6.35 is approximated with two real poles very close to the jw-axis:

$$\frac{\bar{y}_n^*(s^*)}{\delta_f(s^*)} \approx \frac{1}{(s^* + K)^2} \frac{0.9546 \cdot s^{*2} + 1.2582 \cdot s^* + 0.4913}{s^{*2} + 2.1923 \cdot s^* + 1.5797} \tag{6.38}$$

with K = 0.0001. The resulting high DC gain approximates the integrator effect.

The H-infinity controller must balance the tradeoff between three frequency domain criteria: performance weighting, represented by $w_P \cdot S$; control effort, represented by $w_u \cdot KS$; and model uncertainty, represented by $w_I \cdot T$. These three design goals are represented approximately by the minimization of the stacked H-infinity norm below, described in detail in (Skogestad and Postlethwaite, 2000):

$$\|N\|_\infty = \left\| \begin{matrix} w_P \cdot S \\ w_u \cdot KS \\ w_I \cdot T \end{matrix} \right\|. \tag{6.39}$$

Additionally, an exogenous disturbance is added to allow for disturbance rejection in the common case when the steering input may be biased, or where there is a steady disturbance acting on the vehicle such as a road bank angle, for instance the work of Tseng (Tseng, 2001). The control problem is represented diagrammatically in Figures 6.5 and 6.6 below.

While w_I was defined in the previous section, the remaining weights, w_P and w_U, represent design variables. In each of the following weighting functions, M_i is the high frequency gain, A_i is the steady-state gain, and w_{Bi} is the approximate crossover bandwidth. The performance weight is given by:

$$w_P(s^*) = \frac{(1/\sqrt{M_P} \cdot s^* + w_{BP})^2}{(s^* + w_{BP} \cdot \sqrt{A_P})^2} \tag{6.40}$$

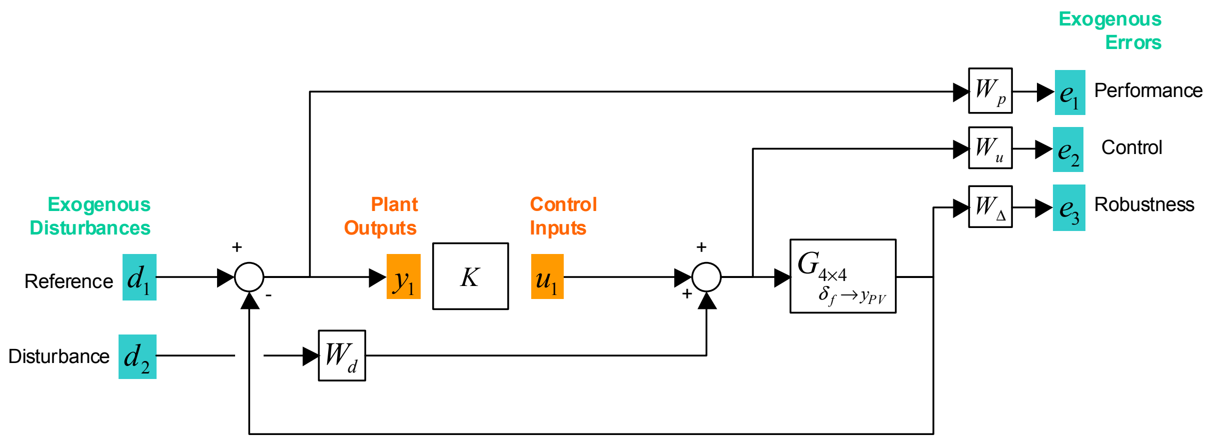


Figure 6.5: Classical form of the mixed-sensitivity H-infinity synthesis problem

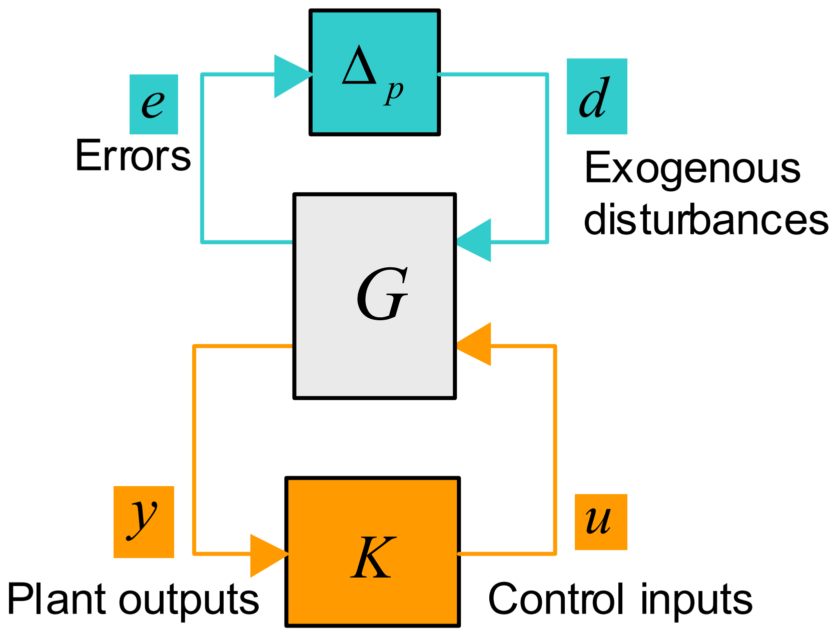


Figure 6.6: Standard form of the H-infinity synthesis problem

With parameters $M_P = 1.5$, with $A_P = 0.01$, and $w_{BP} = 0.1$ rad/sec*. For the control weighting:

$$w_U(s^*) = \frac{(1/\sqrt{M_U} \cdot s^* + w_{BU})^2}{(s^* + w_{BU} \cdot \sqrt{A_U})^2} \qquad (6.41)$$

The control weighting was chosen with $M_P = 1/100$, $A_U = 10$, and $w_{BP} = 200$ rad/sec*. Finally, the disturbance weight is given as:

$$w_D\left(s^*\right)=1 \qquad (6.42)$$

Each of the above performance, control, and disturbance weights were chosen by recursive tuning to maximize system response without violating robustness constraints or control effort usage. These parameters represent the ‘knobs’ of the controller design. While the robustness constraint is fairly strict, the above weights would depend on the selection of the designer.

The H-infinity controller is obtained using standard robust synthesis routines, which solve the control problem by iterating through possible controller representations seeking to minimize the norm of Equation 6.39. A solution was found with a norm of 1.0349, but this solution included a fast pole at s*= -2111. Using model reduction by balanced truncation, the remaining dynamic modes were extracted to produce a controller:

$$\begin{aligned}\frac{U(s^*)}{E(s^*)} &= 6.4274\cdot\frac{\left(s^*+2004\right)}{\left(s^*+158.6\right)}\frac{\left(s^*+10\right)}{\left(s^*+10.35\right)}\frac{\left(s^*+0.1638\right)}{\left(s^*+0.01\right)^2}\\ &\cdot\frac{\left(s^{*2}+0.2421s^*+0.01625\right)}{\left(s^{*2}+1.324s^*+0.5169\right)}\frac{\left(s^{*2}+2.216s^*+1.562\right)}{\left(s^{*2}+15.03s^*+65.06\right)}\end{aligned} \qquad (6.43)$$

Here $E(s^*)$ is the error between the reference signal and previewed feedback. The H-infinity controller synthesis with the previous weights achieved the loop shapes of Figure 6.7, which show that all specifications were met.

The plots of Figure 6.7 reveal that the gamma value larger than unity is due to the control effort exceeding the specified bounds at high frequencies. Because the control bound was implemented primarily to enforce a roll-off shape to the controller rather than a strict numerical bound, this violation of the specified bounds is not of particular concern.

6.4.3 Simulation and Experimental Results

Experimental testing of this H-infinity controller was conducted in both simulation and experimental platforms. The simulation was necessary to represent the full possible range of vehicle plants, while the experimental vehicle is used to introduce real-world plant variations including nonlinearities, unmodeled dynamics, and disturbances that are otherwise ignored in a simulation study.

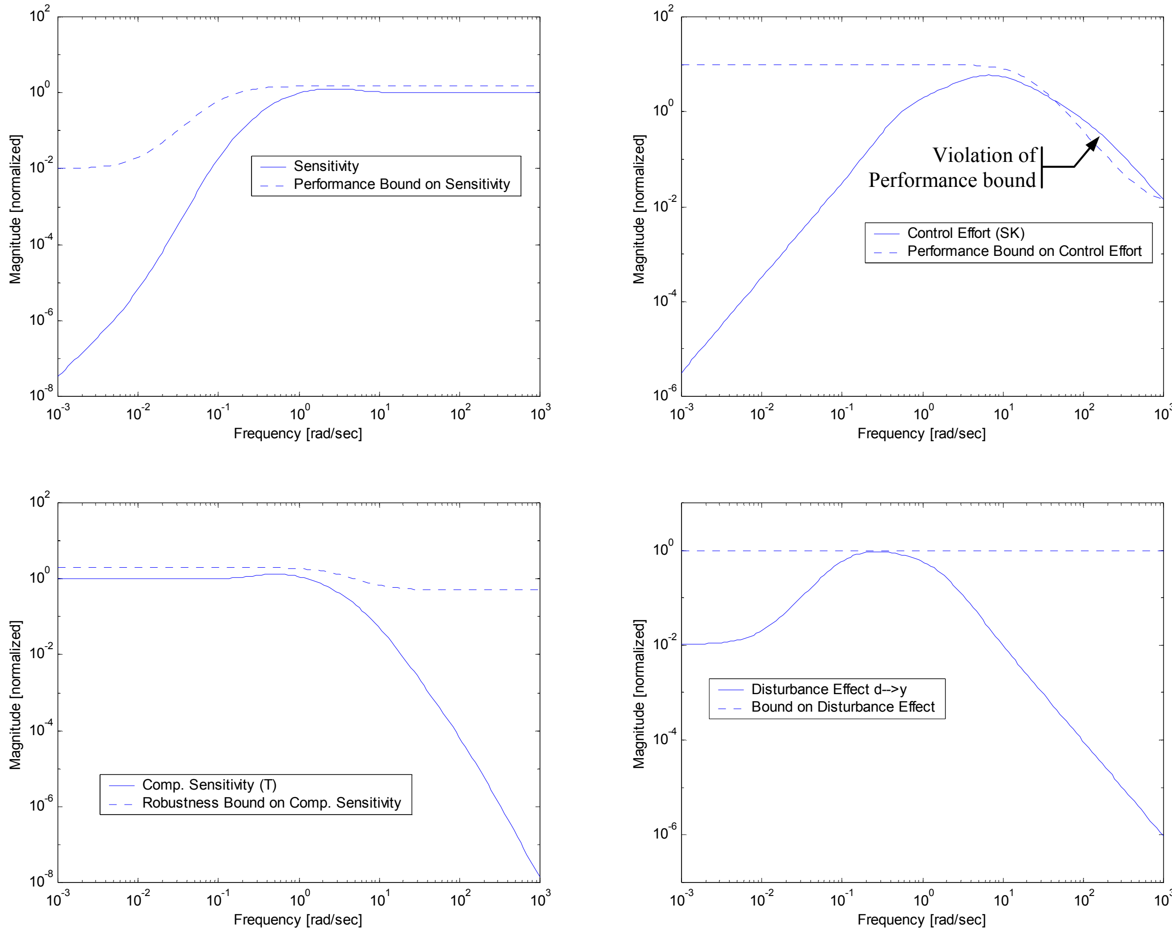


Figure 6.7: Controller loop shapes

The experimental vehicle utilized for testing of the controller is shown in Figure 6.8, and the parameters for this vehicle, given in Table 6.1, were measured from this vehicle using methods described in Chapter 2 and in the Appendix. For the experimental vehicle to operate at a fixed $\pi_3 = 0.5$, it was driven at a speed of 2.95 m/s. This speed approximates an 'average' full-sized vehicle at a speed of 40 mph.

Note that the original generalized controller is designed in dimensionless time and space. For implementation in 'true' space, one must convert the dimensionless controller of 6.43 using the correct time and spatial factors given by the conversions of Chapter 3. The inverse transformations are given from these equations as:

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{A}^* , \qquad \mathbf{B} = \mathbf{S}^{-1}\mathbf{B}^*\mathbf{U}^{-1}$$
$$\mathbf{C} = \mathbf{Y}\mathbf{C}^* , \qquad \mathbf{D} = \mathbf{Y}\mathbf{D}^*\mathbf{U}^{-1} \tag{6.44}$$

To obtain the terms **U** and **Y**, note that the controller input is a tracking error, which has dimensions of length:

$$\mathbf{U} = L \tag{6.45}$$

and the output of the controller is a steering angle, in radians, which is unitless:

$$\mathbf{Y} = 1 \tag{6.46}$$

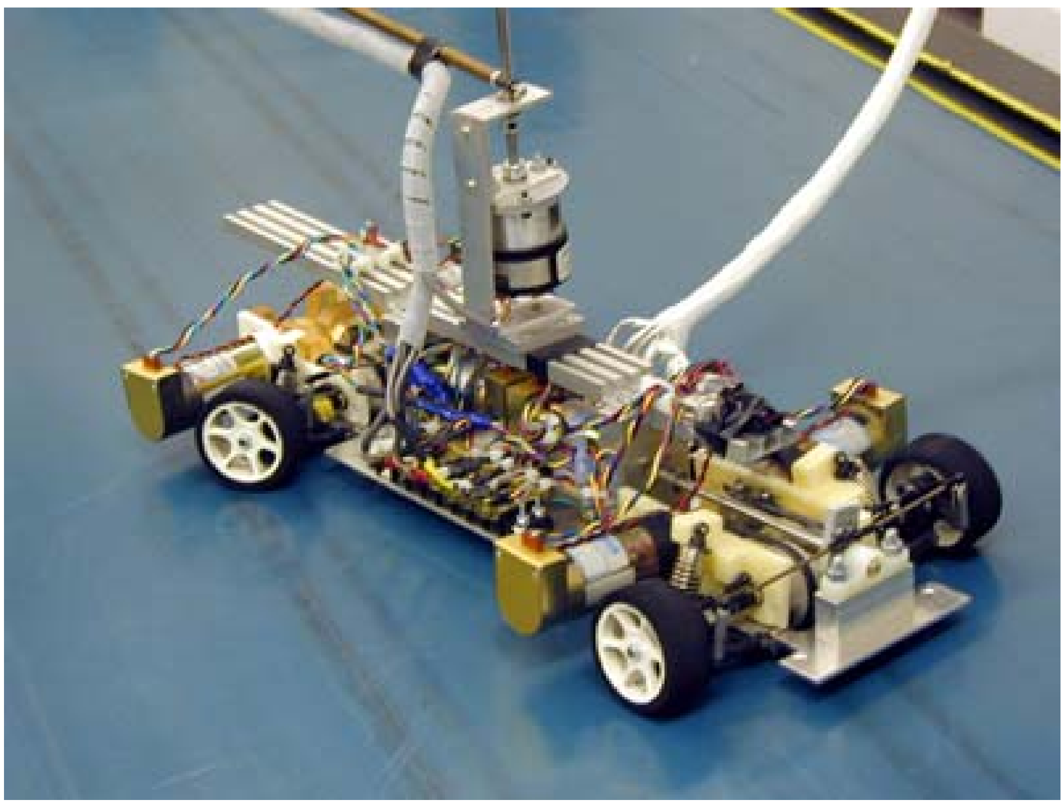


Figure 6.8: Experimental test vehicle

The time conversion, **S**, is given by Equation 6.26. The experimental vehicle parameters are given in Table 6.1. The resulting controller for the scale vehicle would then be:

$$\left.\frac{U(s)}{E(s)}\right|_{ScaleVehicle} = 142.16 \cdot \frac{u^*_{max}}{e^*_{max}} \cdot \frac{(s+1.619e4)}{(s+1281)} \frac{(s+80.79)}{(s+83.58)} \frac{(s+1.323)}{(s+0.08078)^2} \cdot \frac{(s^2 + 1.955s + 1.061)}{(s^2 + 10.69s + 33.72)} \frac{(s^2 + 17.9s + 101.9)}{(s^2 + 121.4s + 4245)} \qquad (6.47)$$

Note that the signal normalizations from Equation 6.33 are included in the conversion above in variable form. An interesting aspect of the conversion is that the vehicle mass is not needed to transform the generalized dimensionless controller to a specific controller for a particular vehicle.

The vehicle responses from the experimental vehicle are shown in Figure 6.9 as the vehicle attempts to track a square wave. The H-infinity controller is slightly underdamped, a result that should be expected in consideration of essentially single-state feedback combined with severe robustness constraints. However, there was no system identification outside of measurement of basic parameters: vehicle mass, vehicle length, vehicle velocity, and tire cornering stiffness. The controller was designed to be robust enough to operate nearly any vehicle, so the effectiveness of the controller on this arbitrary vehicle without identification is not surprising. Note also that the steering angle was limited to 0.5 radians amplitude due to physical limitations on the range in steering angle.

Also shown in Figure 6.9 are the predicted vehicle responses based on a simulation of the linear Bicycle Model (Chapter 2) with the measured parameters of Table 6.1. The close match between measured and predicted results shows a good validation of the model for both controller design and for simulation. Even for the aggressive maneuvers above, a very reasonable match is observed between measured and predicted responses.

Using the bicycle model dynamics of the 40 vehicles in the database (Appendix), the controller of Equation 6.47 was simulated for each vehicle. The results are shown in Figure 6.10 below. The envelope of responses shows a very reasonable set of vehicle responses for a square-wave tracking problem simulating an emergency lane-change maneuver.

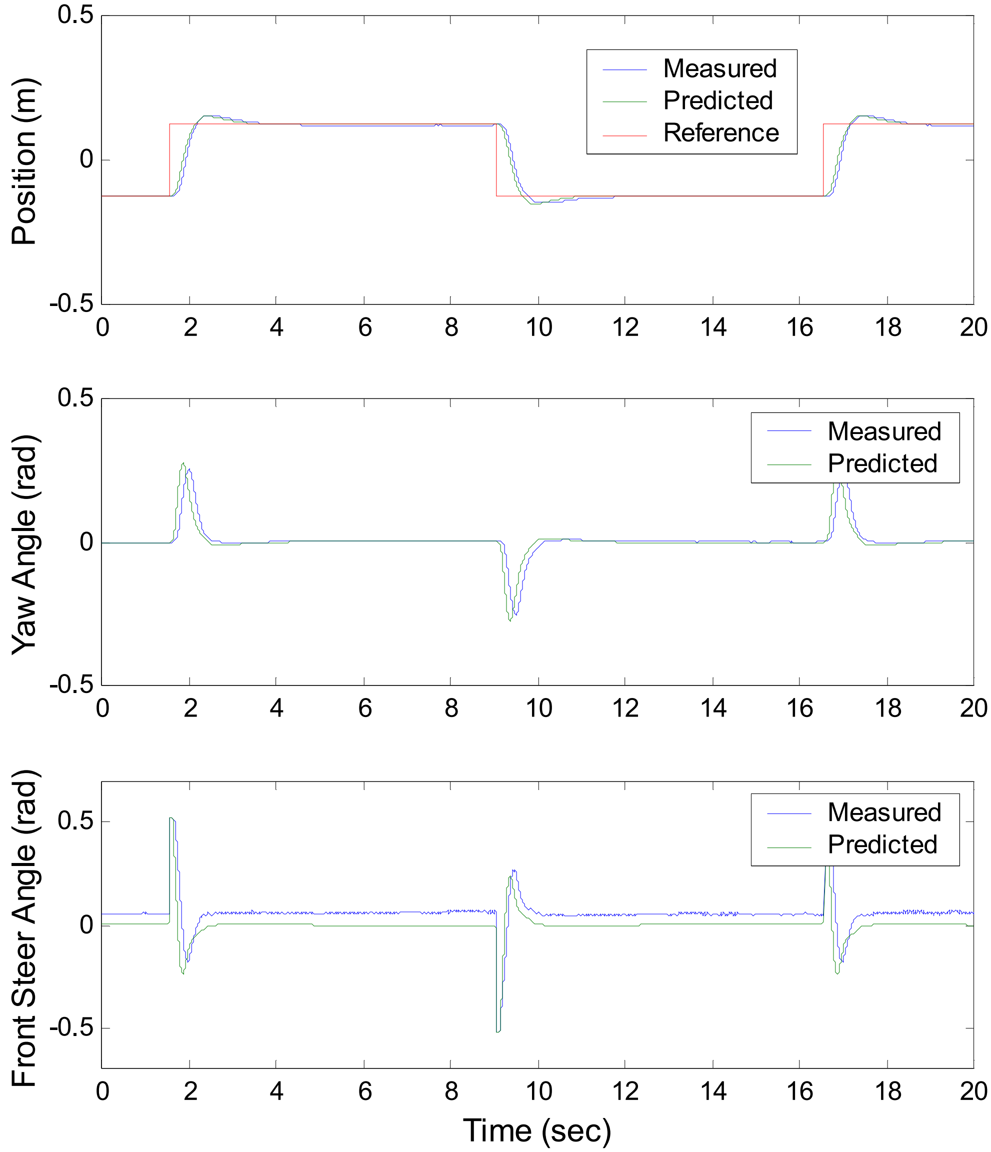


Figure 6.9: Experimental closed loop step responses

Note that each of the vehicles reach different steady-state values because the amplitude of the square wave was made equal to each vehicle's length (in meters), which is different for each vehicle. The larger amplitude responses correspond to longer (and heavier) vehicles. The plots reveal that the larger vehicles are more sluggish in their response, as expected. By nature of using a non-dimensional control formulation, such size effects are implicitly accounted for in the control design.

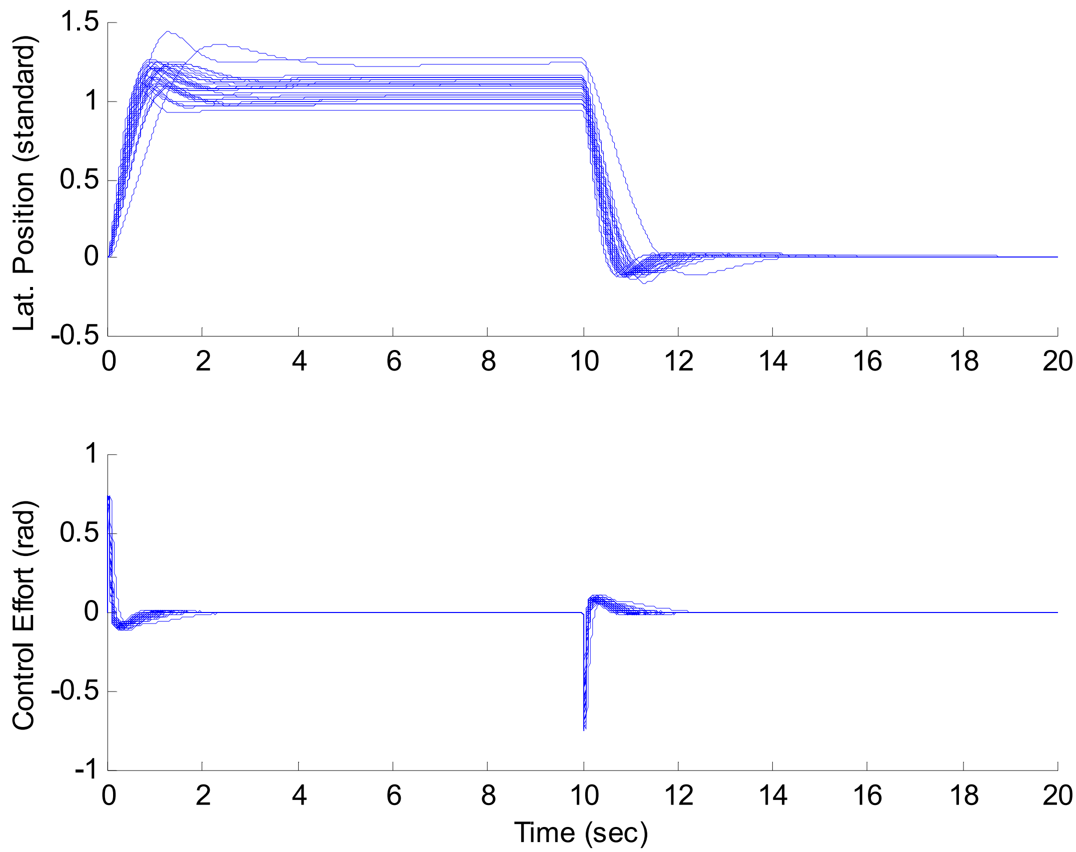


Figure 6.10: Simulated Closed-Loop responses for all vehicles in the database

6.4.4 Remarks

The key factor limiting the controller performance is the tradeoff between small tracking error, good disturbance rejection, and maintaining system robustness. This classic tradeoff between robustness and performance is well known, and the exact choice of values depends on the intent of the control designer. However, it was found that significant tuning could not eliminate the observed overshoot, a fact that is probably due to the fact that the nature of the feedback signal forces a fixed ratio between the gain on lateral position and yaw angle.

Previous work (Chapter 5) found a hard upper limit on generalized vehicle control was set for π_3 values less than 0.2 due to a limit on multiplicative uncertainty of less than unity. It must be mentioned that this limit is probably optimistic. Better controller tuning cannot eliminate the overshoot that appears as a bump in both the sensitivity and complementary sensitivity plots of Figures 6.7. As π_3 is decreased (corresponding to increasing velocities), the robustness bounds to encompass the larger model uncertainty in Figure 6.7 will be more difficult to construct. At some limiting velocity, the peak of the complementary sensitivity plot will exceed robustness limits corresponding to standard vehicle-to-vehicle variations. While this exact limit to velocity was not explored in this work, it should be clear that the robustness violation should occur well before the $\pi_3 = 0.1$ multiplicative uncertainty bound implied by Figure 6.3. The limiting value should be between $\pi_3 = 0.2$ and $\pi_3 = 0.3$, which corresponds to average full-size vehicle speeds ranging from 50 to 60 mph. We can then make the following very general claim:

Robust controller implementations and results for vehicles driving above 50 to 60 mph are not applicable to other vehicles with the same guarantees on stability. Because the multiplicative uncertainty is larger than one for these velocity conditions, the possibility exists that a robust controller on one vehicle will destabilize another vehicle above these speeds.

The critical robustness speed has interesting implications for vehicle control, as other researchers have pointed out that significant modifications to control approaches are needed to achieve high-speed lateral control (Patwardhan, Tan, and Guldner, 1997). In the MIMO control case, the limiting case of Equation 6.36 could be calculated using the maximum singular values of the uncertainty model. In either case, we must conclude that limits do exist to robust, high-speed vehicle control algorithms.

The controller presented in this section addressed controller robustness in a generalized dimensionless framework that brings insight to the feasibility of a robust controller design. By parameterizing plant uncertainty dimensionless, normal distributions were obtained of the plant parameters that defined an average plant. Measured differences between a vehicle database and an average plant motivated a multiplicative uncertainty description. An H-infinity methodology was then presented that utilizes a stacked sensitivity approach. The controller results were demonstrated both in simulation and on a research vehicle. While this approach achieved robust control, it revealed limits to extending the approach with respect to vehicle velocity scheduling. Specific velocity ranges exist above which generalized robust control is no longer achievable.

6.5 Contributions of This Chapter

The primary contributions of this chapter are as follows, numbered by relation to corresponding sections of the chapter:

(1) A history of dimensional analysis as applied to generalized dynamic model representations is presented, and methodologies found to be useful by other authors are discussed

(2) Generalized forms of system dynamic representations and methods are discussed,

   a. Direct methods of using dimensional analysis on a governing differential equation are shown to yield results identical to parametric approaches

   b. The number of pi-groups associated with arbitrary homogenous, linear differential equations is shown to be equal to the number of terms minus two.

   c. The dimensionless form of the traditional form of Lagrange dynamics is shown to require similarity of a Froude-number if different Lagrangian systems dynamics are required to be equivalent.

(3) To illustrate a dimensionless robust controller design based on frequency-domain bounds (rather than parameter bounds in the earlier chapter), a robust vehicle controller design is presented.

   a. A database of dimensionless parameters for vehicle dynamics was created from the literature.

   b. The nominal system was obtained by observing the peak of the relative frequency distributions of the dimensionless parameters (from Chapter 4).

   c. A frequency response of the multiplicative model error was obtained by comparing each vehicle in the database to the nominal system

   d. An $\mathcal{H}_\infty$ controller design was found based on the above uncertainty bounds.

   e. The controller was transformed back into the physical domain via simple transforms.

   f. An experimental implementation was presented demonstrating the controller on the vehicle testbed of Chapter 2.

   g. Bounds on the above error were shown to be nearly constant for a fixed speed, and the multiplicative uncertainty rises above one for vehicles at speeds higher than approximately 60 miles per hour. In agreement with the result of Chapter 5, a generalized vehicle design was found to be infeasible at high speeds.

6.6 References

1. Alleyne, Andrew. "A Comparison of Alternative Obstacle Avoidance Strategies for Vehicle Control." Vehicle System Dynamics 27 (1997): 371-92.
2. Birkhoff, G. "Dimensional Analysis of Partial Differential Equations." Elec. Engineering 67 (1948): 1185.
3. Duncan, W. J. Physical Similarity and Dimensional Analysis: An Elementary Treatise. London: Edward Arnold & Co., 1953.
4. Guldner, J., H. S. Tan, and S. Patwardhan. "Analysis of Automatic Steering Control for Highway Vehicles With Look-Down Lateral Reference Systems." Vehicle System Dynamics 26.4 (1996): 243-69.
5. Kline, Stephen J. Similitude and Approximation Theory. 1st ed. New York: McGraw-Hill, 1965.
6. Patwardhan, Satyajit, Han-Shue Tan, and Jurgen Guldner. "A General Framework for Automatic Steering Control: System Analysis." Proceedings of the American Control Conference: 1997. 1598-602.
7. Peng, H., and M. Tomizuka. "Preview Control for Vehicle Lateral Guidance in Highway Automation." ASME Journal of Dynamic Systems, Measurement and Control 115. December (1993): 679-86.
8. Ruark, A. E. "Inspectional Analysis: A Method Which Supplements Dimensional Analysis." J. Elisha Mitchell Sci. Soc. 51 (1935): 127-33.
9. Skogestad, Sigurd, and Ian Postlethwaite. Multivariable Feedback Control, Analysis and Design. 1996. Baffins Lane, Chichester England: John Wiley & Sons Ltd., 2000.
10. Taylor, Edward S. Dimensional Analysis for Engineers. Oxford: Clarendon Press, 1974.
11. Tseng, H. E. "Dynamic Estimation of Road Bank Angle." Vehicle System Dynamics 36 (2001): 307-28.

Chapter 7
Conclusions and Future Work

7.1 Summary of Chapter Results

To summarize the key points emphasized throughout this thesis, the main results are listed below in relation to the chapters in which they were made:

7.1.1 Chapter 2: Vehicle Control

(1) Introduce vehicle notation and governing dynamic models for chassis motion at highway speeds.

(2) Illustrate that size-independent controller designs and model comparisons are necessary for certain control problems.

7.1.2 Chapter 3: Dimensional Analysis

(1) Demonstrate that there is a rich history of dimensional analysis, and that the key contributors to this field include many of the greatest scientists, engineers, and mathematicians of humanity...

(2) Introduce basic notions of physical dimensions and their use in basic measurements.

(3) Present the basic unit systems in use today and dispel the notion that any one system may be 'superior' to another.

(4) Demonstrate how to convert between different dimensioning systems (i.e. unit systems) and discuss how the use of a unit system is generally based on an assumption of the

Absolute Significance of Relative Magnitude, and that some 'measurement' systems violate this assumption.

(5) Argue that the mathematical use of dimensioned quantities requires an implicit, structured, and carefully constrained set of mathematical operations that are dependent on the dimensions of the arguments.

(6) Argue that mathematical operations on dimensioned quantities is best represented by operations on an ordered pair consisting of a real term and a vector quantity of rational numbers. The use of such mathematical orderings:

   a. Imposes a sign-symmetry on all physical descriptions.
   b. Shows that dimensioned mathematics is not closed under addition
   c. Constrains arguments to most mathematical functions
   d. Is extendable to specialized forms of vectors and matrices

(7) Argue that dimensional constraints of the vector form above limit allowable forms of physical equations. Specifically,

   a. There are a limited number of possible variable combinations for a given problem that can satisfy unit constraints necessary in the equation solution.
   b. The possible variable combinations from which a solution set must exist can be generalized to a set of linear dimension-vector equations.
   c. These dimension-vector equations are always under-determined.
   d. A reparameterization to dimensionless parameter forms is obtainable by a partial-solution to the dimension-vector equations and always reduces the number of parameters in an equation description

(8) Introduce basic dimensionless representations of system equations. These forms:

   a. Are derivable either by direct variable parameterizations or by simple state-substitutions in state-space forms combined with a temporal renormalization.
   b. Directly generalize the results of numerical balancing and normalization methods generally used in numerical analysis techniques.
   c. Demonstrate (by example) that similar pole locations do not guarantee dynamic similitude (This topic is of such importance that it is discussed in great detail in following chapter).

7.1.3 Chapter 4: Sensitivity Analysis

(1) The discussion of parametric uncertainty in the remainder of the chapter (and thesis) shares analysis techniques and results with the analysis of system behavior in the presence of disturbances. In many (perhaps most) circumstances, one can represent parametric variations with exogenous disturbances, or in the reverse argument, one may often represent true disturbances via exogenous parametric variations.

(2) There is a rich history and field of study related to sensitivity analysis with strong ties to the University of Illinois. The historical sensitivity notion of the Miller-Murray classification system delimits major areas of focus within each chapter in this thesis.

(3) Euler's Homogenous Function Theorem (EHF Theorem) was used to prove the Pi-Theorem of the last chapter. This theorem predicts equations for invariant Bode sensitivity for a problem representation. Specifically:

   a. Every dimensioned system representation contains at least one and usually multiple subspaces of sensitivity invariance.
   b. The sensitivity invariance is always described by a set of linear equations.
   c. For a system of n parameters spanning d physical dimensions, there will be (n-d) equations of sensitivity invariance.

(4) Examples illustrate that:

   a. System Bode sensitivities to parameter perturbations are usually coupled to sensitivity to state variables (i.e. state-derivatives) by sensitivity invariance equations.
   b. The sensitivity of mappings (equations) consisting only of one dimensionless parameter are often determinable without knowledge of the system dynamics.
   c. The sensitivity invariants apply in static mappings, time-domain dynamic representations, or frequency-domain representations of systems in open or closed feedback loops.
   d. Numerical or algebraic calculation of the sensitivities can be used to verify invariance relationships

(5) The equations for sensitivity invariants can be eliminated by a reparameterization of the system equations using parameter ratios.

a. Each system reparameterization to eliminate one sensitivity invariance equation eliminates one dimensioned parameter from the governing equation
b. In the limiting case, the form that will eliminate all sensitivity equations is dimensionless and will have eliminated (n-d) parameters from the system representation
c. The equation forms that eliminates sensitivity invariants is identical to the equation forms required by dimensional analysis to make claims about dimensional similitude

(6) Classical (dimensioned) forms of system representation are inappropriate for comparison of systems and for generalizing controller designs. Specifically:
a. Numerically equivalent differential equations (transfer functions, state-space, etc.) representations do not imply equivalent system sensitivity, and hence controller design.
b. In the reverse argument, systems that are numerically different may indeed be equivalent with respect to system sensitivity, and hence controller design.
c. The notion of dimensionless representations corrects the above flaws associated with sensitivity measures and once again allows for numerical comparisons to be made between systems.

(7) The notion of system equivalence in a dimensionless framework allows for direct and numerically measurable notions of system equivalence. Specific advantages include the following:
a. The notion of system-to-system equivalence generalizes to system-class notions of system behavior. A single 'class' of systems tend to cluster into localized regions of the dimensionless pi-space
b. Systems within a similar model class will share sensitivity invariance equations due to optimization or design constraints. These constraints generate well-defined power-law relationships between the model parameters.
c. A broad range of systems spanning mechanical and biological examples appear to exhibit mathematical properties associated with very localized dimensionless 'classes' of system representations.

d. The nominal system within a class as well as the range of behavior *over* a class is easily and numerically measurable in the dimensionless parameter domain, where it may not be obvious (or defined) in standard physical domains. The vehicle dynamics example was presented and a nominal (average) vehicle parameter was calculated from a dimensionless viewpoint.

7.1.4 Chapter 5: Parametric Methods of Dimensional Analysis in Control

(1) The use of Dimensional Analysis simplified stability analysis of a system. Specifically,

a. A nonlinear result of the Routh stability criteria for vehicle dynamics consisting of seven parameters was simplified to a linear criteria for stability consisting of two parameters.

b. The above line representation correctly characterized the observed stability of actual vehicles; vehicles farthest from the above line were most stable, while vehicles closest to the above line were the least stable.

(2) The use of dimensionless parameters and dimensionless model representation made possible a complex model reduction problem of a heating and cooling systems. Namely,

a. Singular perturbation and other model-reduction techniques were analyzed from a dimensional standpoint. Based on matrix conditions, many (but not all) of the standard techniques were found to be valid only for dimensionless model representations.

b. The methods of dimensional analysis are found to be implicitly used by many authors, but in an incomplete manner that does not eliminate all sensitivity invariants and introduces new problem parameters.

(3) A model-order reduction of a analytically derived cooling system model is presented. The goal of the reduction was to preserve physical meaning, and a dimensionless representation specifically allowed comparison of systems with different state definitions.

(4) The use of dimensionless parameters is shown to *significantly* reduce the complexity of control problems that may be gain scheduled with respect to more than one parameter

a. A vehicle example is presented with the dual parameter variations of road friction and vehicle velocity are shown to be representable by a single gain-scheduled parameter
b. A generalized statement is made regarding the maximum number of parameters that can be coupled in such a dimensionless parameter approach. The number increases as the number of physical dimensions plus one.
c. An example is presented of a classical gantry system where a system of four parameters (two of them scheduling variables) is reduced to a dimensionless system of one parameter.

(5) To illustrate a primary advantage of dimensionless representations, a robust vehicle controller design is presented where model perturbations are modeled via dimensionless parameters.
a. A database of dimensionless parameters for vehicle dynamics was created from the literature.
b. The database revealed that the dimensionless system parameters are very interdependent; they tend to span the dimensionless pi-space almost exactly as a line (rather than a blob). This interdependence is conjectured to be due to common design constraints and a high level of design optimization.
c. A representation is chosen using perturbations about the nominal line through pi-space. The system dependence on the pi-parameters is approximated by an affine (linear) representation where the system matrices are linearly dependent on the model parameters.
d. The model perturbations were made wide enough to capture every vehicle in the database (approximately every vehicle in production)
e. To illustrate an ability to design controllers in a dimensionless framework, an LMI-based design found a solution to the control problem based on the affine representation, but also demonstrated that robust control over a wide variation in speeds (and road frictions) is not feasible.
f. The controller was transformed back into the physical domain via simple transforms.

g. An experimental implementation was presented demonstrating the controller on the vehicle testbed of Chapter 2.

7.1.5 Chapter 6: Dynamic Methods of Dimensional Analysis in Control

(9) A history of dimensional analysis as applied to generalized dynamic model representations is presented, and methodologies found to be useful by other authors are discussed

(10) Generalized forms of system dynamic representations and methods are discussed,

a. Direct methods of using dimensional analysis on a governing differential equation are shown to yield results identical to parametric approaches
b. The number of pi-groups associated with arbitrary homogenous, linear differential equations is shown to be equal to the number of terms minus two.
c. The dimensionless form of the traditional form of Lagrange dynamics is shown to require similarity of a Froude-number if different Lagrangian systems dynamics are required to be equivalent.

(11) To illustrate a dimensionless robust controller design based on frequency-domain bounds (rather than parameter bounds in the earlier chapter), a robust vehicle controller design is presented.

a. A database of dimensionless parameters for vehicle dynamics was created from the literature.
b. The nominal system was obtained by observing the peak of the relative frequency distributions of the dimensionless parameters (from Chapter 4).
c. A frequency response of the multiplicative model error was obtained by comparing each vehicle in the database to the nominal system
d. An $\mathcal{H}_\infty$ controller design was found based on the above uncertainty bounds.
e. The controller was transformed back into the physical domain via simple transforms.
f. An experimental implementation was presented demonstrating the controller on the vehicle testbed of Chapter 2.

g. Bounds on the above error were shown to be nearly constant for a fixed speed, and the multiplicative uncertainty rises above one for vehicles at speeds higher than approximately 60 miles per hour. In agreement with the result of Chapter 5, a generalized vehicle design was found to be infeasible at high speeds.

7.2 Conclusions

In addition to the point-by-point results discussed above, overarching conclusions can be made regarding the synthesis of dimensional analysis and control theory. Specifically, this thesis argues that:

(1) Dimensional analysis reveals invariant sensitivity equations that exist for every dimensioned physical equation representation of a system,

(2) Traditional, dimensioned forms of system equations may be inadequate because they fail to address sensitivity invariance.

(3) Sensitivity invariant equations can be eliminated by a dimensionless reparameterization of the system model.

(4) Analysis of a dimensionless model measured over many different physical systems can be used to formalize the concept of a system class. Distributions of similar models within a given class show a defined structure that is useful for characterizing groups of systems for a robust controller design.

(5) Controller designs in the dimensionless space are easily generalized to standard forms and have specific advantages over traditional forms in the areas of model reduction, gain scheduling, and robust controller implementation.

In summary, the use of dimensional analysis has implications on control methodology that are more profound than spatial and temporal scaling rescaling. The biggest contribution of this thesis is the presentation of dimensional similitude in a system-sensitivity context that extends to nearly every field of control. The sensitivity invariance concepts extends and generalizes specific results obtained on system sensitivity invariants dating back to the mid-1960's.

There is a large amount of future work remaining in the study of dimensional analysis and control systems. The following sections address general areas that were investigated in the creation of this thesis but whose results were not full enough to meet the satisfaction of the author. In many cases, very well-developed examples of dimensionless implementations have already been investigated and are included in the discussion of .

7.3 Future Work Related to Parameter Reduction

Many of the results of the thesis were based on the observation that dimensionless systems exhibit reduced number of parameters in the system model. There remain many key questions regarding this type of system reparameterization in a controls context.

7.3.1 Converting an LPV-LTV System to a Dimensionless LTI Model

If one examines a Linear, Parameter-Varying (LPV) system representation from a dimensional analysis viewpoint, often one can couple the time-varying variable with the unit of time via a parameter substitution. In this case, a linear-time-invariant system is often produced. Obviously, such reparameterization would be beneficial, but questions remain in the author's opinion on the practicality and feasibility of such an approach. The question arose in this thesis in the area of gain-scheduling. It is easy to construct examples where the gain-scheduled physical parameters may be rapidly changing in time yet the pi-parameters of the problem representation are constant. Additional investigation is obviously warranted to discern the theoretical meaning of such behavior and how pi-parameterizations may be explicitly constructed to map LPV systems to LTI systems.

7.3.2 Adaptive Identification of Unknown Parameters

Chapter 4 introduced the notion of sensitivity invariant subspaces. This notion would be especially useful in Adaptive Control, which is often based on using gradient or related techniques to generate parameter estimates. The relationship between sensitivity analysis and adaptive system identification and control is described in several good references, see (Eslami, 1994; Sastry, 1999; Slotine and Li, 1991). However, traditional treatments of adaptive methods do not discuss sensitivity invariants. To include these invariants in the analysis, there are two general options:

(1) Utilize a dimensionless pi-based model representation, thus eliminating the invariant subspaces

(2) Recognize the sensitivity invariants, and use them to solve for or refine parameter gradient estimates

In both cases, given an identification model consisting of N_V dimensioned variables spanning N_d dimensions, the number of parameter estimates may be reduced by a factor of $N_V - N_d$ via dimensional analysis.

The first approach has historically been especially useful for static mappings. For instance, if the sensitivity to a model with respect to a certain parameter is low, then the partial derivative with respect to that parameter within the system output will be nearly zero. A method for checking for this condition is a simple graphical analysis. If one has access to data spanning a large range of parameter variations, an observation that the dependent parameters are measurements are independent of the other will justify the elimination certain variables from the equation. This is illustrated below via an interesting example of identification a static mapping. By plotting a dependent pi-parameter with respect to an independent pi-parameter (on the x-axis), the irrelevant dependent variable will appear as a horizontal line.

Example: Graphical Interpretation of Kepler's Data. We reconsider Kepler's Third Law of planetary motion discussed in Chapter 3, specifically from the standpoint data-interpretation. If we plot the data for the 9 planets using the two pi values determined previously:

$$\pi_1 = T \cdot \sqrt{\frac{M_S \cdot k}{a^3}}, \pi_1 = \frac{M_P}{M_S} \qquad (7.1)$$

we note that π_1 is the dependent variable if we are attempting to determine the period of the planets. The resulting plot of the two variables is given below in Figure 7.1.

If we note the scale on the y-axis, we see that the dependent pi parameter is constant over a very large range in the independent parameter. With some certainty, we may conclude that a change in the independent parameter does not appear to affect the independent parameter in any significant fashion. This is a numerical way of stating that the model sensitivity with respect to pi 1 is very small or zero. If this is true, then this parameter may be eliminated from the system of equations as it is simply a constant. We therefore arrive at the same conclusion as Kepler, who stated that the period of the planets does not depend on the mass of the planet under consideration.

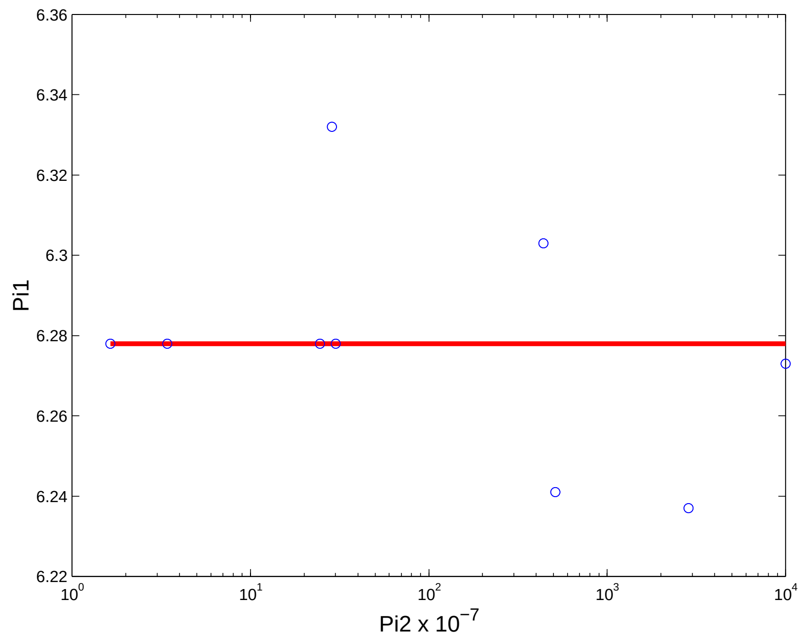


Figure 7.1: Pi Interdependence for Kepler's Third Law

From an adaptive control standpoint, an additional design-level degree of freedom arises in a dimensionless reparameterization. Specifically, one can often 'choose' the location of certain dimensioned variables within pi-parameters by careful construction of the pi-parameters. If we consider again the general case of a model where more than one pi parameter is active in the model, then there may be some question regarding the method to investigate the governing law. Experimentalists familiar with the problem generally suggest the following rules (see Duncan, p. 58):

1. The dependent variable, parameter, or output measure of interest should occur in one and only one pi parameter, a.k.a. the dependent parameter.
2. The remaining variables occurring in the dependent pi parameter should be chosen so that the dependent pi-parameter should remain as constant as possible over the range of the remaining pi-parameters.

Both of the above heuristic rules (which are in general use in the field of control) should be derivable from sensitivity considerations. The first consideration is obviously meant to simplify the form of the partial derivatives with respect to the output variable (see the sensitivity calculations of Chapter 4), while the second condition is meant to limit the change of the dependent variable so that sensitivity calculations with respect to remaining pi-parameters are well-posed. However, the mathematical justifications for such empirical rules, while they certainly must exist, have not yet been encountered by this investigator.

In the usual case of adaptive control or system identification, some of the parameters are usually well-known but the system is *nonlinear* in the remaining parameters. In many cases, it is also often possible to reparameterize the pi-parameters such that nonlinear parameter dependence can be eliminated or reduced.

7.3.3 Pi-Parameter Magnitudes as a Measure of System Sensitivity

For a long period of history, many authors have noted that the pi-values of nearly all equations are neither very large nor very small, i.e. of o(1) in the order notation (numerically, nearly all pi values are in the range of 10^{-4} to 10^{4}. Indeed, one can often eliminate variables from the model representation by a simple claim that the magnitude of the pi variable is either:

(1) So small as to require an exceedingly large coefficient (or input signal) to affect the system or,

(2) So large that any non-zero sensitivity of the system with respect to the parameter would cause the system response to be dominated solely by the one parameter

A more formal sensitivity reasoning behind the above statement has not yet been developed for control systems (or any systems observed by the author), and a better development of the above statement would be greatly useful in the framework of generalized model reduction. Specifically, extensions to frequency-domain ranges of parameter influence are needed. For a survey of the subject, one should see the review paper by Bond written in the late 1920's yet still an interesting area of research to this day (Bond, 1929).

7.3.4 Terminal Conditions Insensitivity and Dimensional Analysis

Historically, the study of guidance theory and its associated theoretical underpinnings was a much more active area of research. The relationship of guidance theory to sensitivity analysis is well developed, and relationships between this theory and the sensitivity concepts of

Perkins and Cruz and the reproducibility concepts of Brockett and Mesarovic are well established (Eslami, 1994).

An important result of previous work was the notion of terminal conditions insensitivity (TCI), which addresses the feasibility of a controllable system to arrive at a terminal condition in the presence of parametric variations. Essentially, the goal of guidance is to reach precisely a target by any appropriate means. One of the necessary system conditions for the parameter variation to have zero effect on terminal conditions is that the system be completely controllable and that the number of varying parameters is equal to the number of system control inputs.

Obviously a dimensional analysis can provide great benefit to TCI theory by reducing the number of system parameters. For systems where the number of parameter variations exceeds the number of control inputs by a number less than the dimensional span, then a dimensionless reparameterization should be able to recover the system conditions required above. Additional investigation into this problem is certainly warranted.

7.4 Future Work on Nonlinear Dimensional Analysis

The original use of dimensional analysis, and indeed the reason for its original development, was the study of nonlinear system behavior. Much of the thesis discussion on the usage of dimensional analysis in control theory has focused on linear systems. There has been some implicit extension of concepts to the control of nonlinear systems, however a well-developed analysis is still lacking. In this chapter, it is argued that most of the previous conclusions shown for linear systems should extend equally to nonlinear systems under certain basic assumptions regarding the existence of sensitivity partial derivatives (i.e. smoothness conditions).

7.4.1 Dimensional Analysis on Localized Nonlinearities

In many control loops, the system might be considered a LTI dynamic system except that a localized nonlinearity exists in the control loop. This localized nonlinearity may be a saturation term on a control input, or a rate-limit due to maximum actuator speeds, or friction term. Although the discussion below is not yet generalizable, it should motivate consideration into dimensional analysis of such problems.

For illustration, consider a motor with a localized nonlinearity in the control loop as shown below. The nonlinearity is assumed to be a rate limit of slope limit, N.

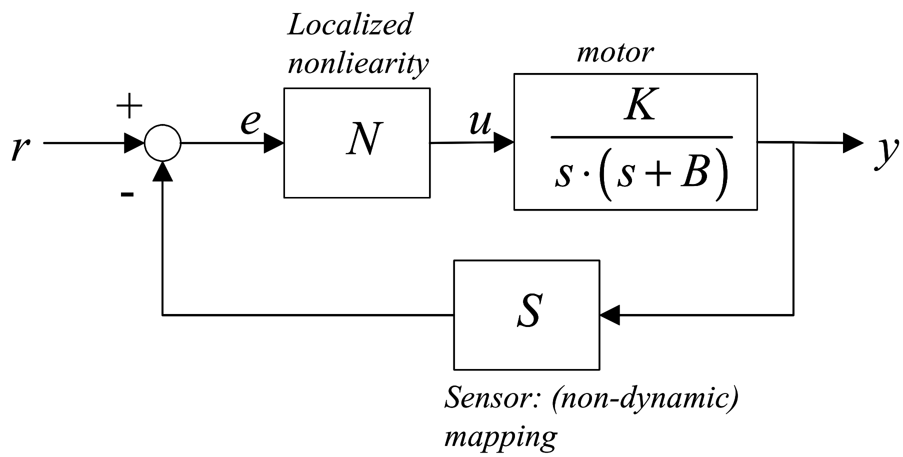


Figure 7.2: A motor control loop with a localized nonlinearity

The system model parameters are described below with their dimensions in control-input (u), system output (y) and Laplace-variable (w… to prevent confusion with seconds) units.

Table 7.1: Physical meaning and units of parameters for the rate-limited motor example

Variable	Symbol	Dimension
Motor gain	K	$\mathrm{y \cdot u \cdot w^2}$
Motor time constant	B	w
Sensor gain	H	$\mathrm{y^{-1} \cdot u}$
Rate limit slope	N	$\mathrm{u \cdot w}$
Reference signal	r	u
Output signal	y	y
Time	t	$\mathrm{w^{-1}}$

Without demonstration of the calculations, a complete set of pi-parameters of the linear system without the rate limit are given by:

$$\pi_1 = \frac{y \cdot B^2}{r \cdot K}, \pi_2 = \frac{H \cdot K}{B}, \pi_3 = t \cdot B \qquad (7.2)$$

and a complete set of the pi-parameters of the nonlinear system with the rate-limit are given by:

$$\pi_1 = \frac{y \cdot B^2}{r \cdot K}, \pi_2 = \frac{H \cdot K}{B}, \pi_3 = t \cdot B, \pi_4 = \frac{N}{r \cdot B}, \tag{7.3}$$

Note that the nonlinear system, by adding a term described by one parameter, simply added another pi-value to the dimensionless representation.

We now explore whether equivalence of the above parameters preserves a controller analysis of the nonlinear system. Using an example, we consider four test cases:

Table 7.2: Test parameters for the rate-limited motor example

Variable	Symbol	Test 1	Test 2	Test 3	Test 4
Motor gain	K	1	2	1	1/2
Motor time constant	B	1	1	2	1/2
Sensor gain	H	1	1/2	4	1/2
Rate limit slope	N	1	1/2	8	1/4
Reference signal amplitude	r	1	1/2	4	1/2

The reader may confirm that each of the above test cases shares identical pi-parameters. A simulation of the systems with the four cases of the above parameters shows very different (but scale-similar) behavior. This can be seen in the responses shown in Figure 7.3. The plots affirm the notion of system equality developed in Chapter 4 of the thesis. The reader is encouraged to examine the values of Table 7.2 to confirm that the choice of input reference amplitude, motor gain, etc. to achieve matching of control responses is not a trivial task. The use of dimensional analysis clearly provides utility in generalizing the control results.

A future avenue of research would be to develop control strategies that allow generalized solutions to systems with localizable nonlinearities. The generality of the approach would parallel that of more traditional, describing function analysis. One would generate a solution to a general, pi-parameterized problem representation (as in the example above), and future control designers could then dimensionally scale the results to their particular problem, as appropriate.

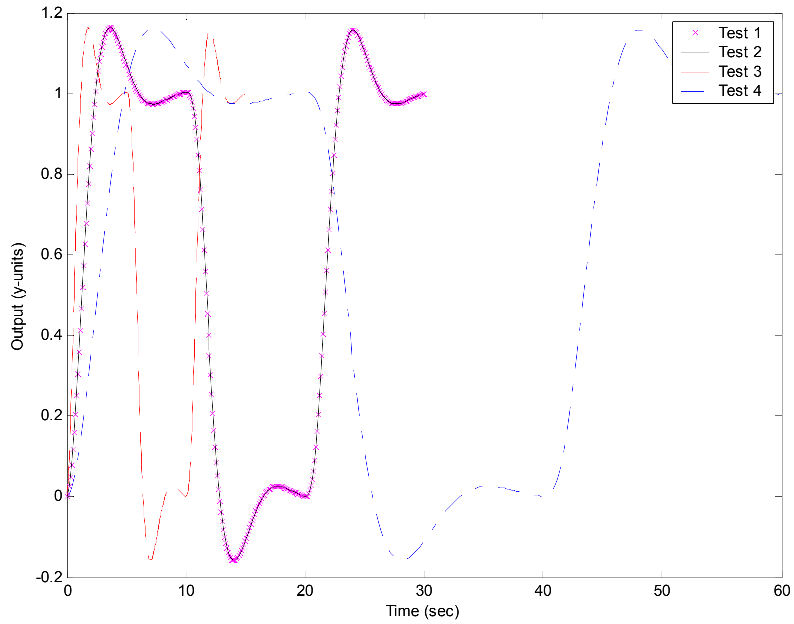


Figure 7.3: Time responses of the rate-limited system

However, in the dimensionless domain, the responses are identical:

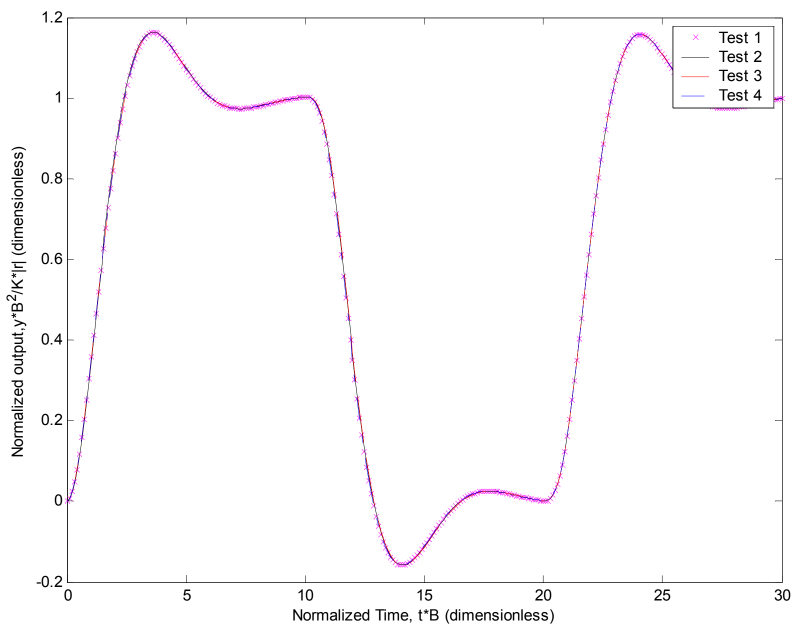


Figure 7.4: Dimensionless time responses of the rate-limited system

7.4.2 Dimensional Analysis on Feedback Linearizable Nonlinear Systems

Many nonlinear systems have a model dependence on the nonlinearities that don't fit in the previous framework developed for localizable nonlinearities in the previous subsection. We consider these types of systems in this section, specifically focusing on an example of the well-known SCARA (Selective Compliance Assembly Robot Arm) robot example to motivate the discussion (Spong and Vidyasagar, 1989; Slotine and Li, 1991). To present the dynamics of the SCARA robot, consider the diagram shown in the figure below:

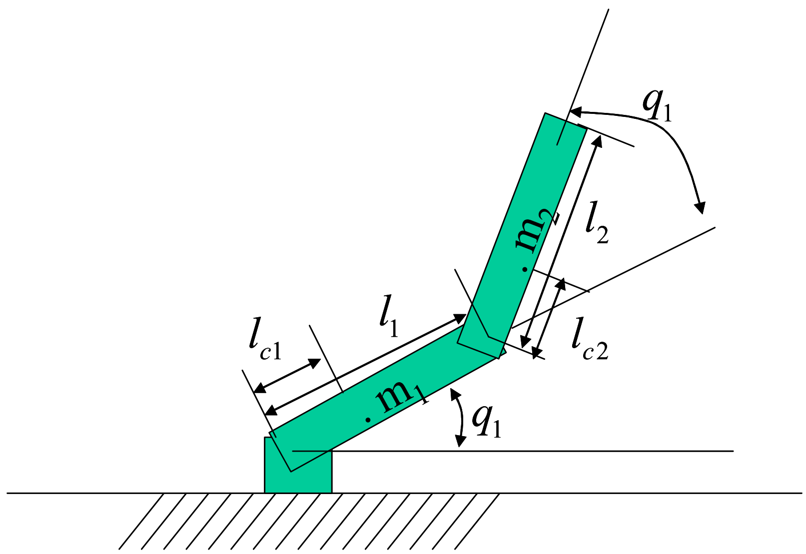


Figure 7.5: SCARA robot arm representation

Using well-known Lagrangian equations, the dynamic equations of the robot can be derived as:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} h \cdot \dot{q}_2 & h \cdot \dot{q}_1 + h \cdot \dot{q}_2 \\ -h \cdot \dot{q}_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{7.4}$$

with $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ being the two joint angles, $\begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}^T$ being the two joint inputs, and:

$$\begin{aligned} H_{11} &= m_1 \cdot l_{c_1}^2 + I_1 + m_2 \cdot \left(l_1^2 + l_{c_2}^2 + 2 \cdot l_1 \cdot l_{c_2} \cdot \cos(q_2) \right) + I_2 \\ H_{22} &= m_2 \cdot l_{c_2}^2 + I_2 \\ H_{12} &= H_{21} = m_2 \cdot l_1 \cdot l_{c_2} \cdot \cos(q_2) + m_2 \cdot l_{c_2}^2 + I_2 \end{aligned} \tag{7.5}$$

$$h = m_2 \cdot l_1 \cdot l_{c_2} \cdot \sin(q_2) \tag{7.6}$$

$$g_1 = m_1 \cdot l_{c_1} \cdot g \cdot \cos(q_1) + m_2 \cdot g \cdot \left(l_{c_2} \cdot \cos(q_1 + q_2) + l_1 \cdot \cos(q_1)\right)$$
$$g_2 = m_2 \cdot l_{c_2} \cdot g \cdot \cos(q_1 + q_2) \tag{7.7}$$

This can be compactly represented in vector form as:

$$\mathbf{H}(\mathbf{q}) \cdot \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} \tag{7.8}$$

One of the more methods to control this system is to use a methods of feedback linearization commonly known as the "computed torque" method (Spong and Vidyasagar, 1989; Slotine and Li, 1991). This method uses a control law that includes a feed-forward term that cancels the nonlinear terms:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -h \cdot \dot{q}_2 & -h \cdot \dot{q}_1 - h \cdot \dot{q}_2 \\ h \cdot \dot{q}_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{7.9}$$

The resulting system representation becomes simply a pure-integrator, linear system:

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.10}$$

One can then design a linear control law in the linear domain to guarantee certain convergence properties. For this simple study, a simple proportional-derivative control law is considered.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = K_p \cdot \begin{bmatrix} q_{1r} - q_1 \\ q_{1r} - q_1 \end{bmatrix} + K_d \cdot \frac{d}{dt} \begin{bmatrix} q_{1r} - q_1 \\ q_{1r} - q_1 \end{bmatrix} \tag{7.11}$$

The performance of this controller will be considered after presenting a dimensional analysis study of the problem.

To determine the invariant sensitivity equations for this system/controller combination, dimensional analysis is used. To begin, we can utilize any method to form dimensionless parameters, but due to the complexity of the model it is obviously easiest to directly calculate the pi-values for the model. To perform this calculation, we review the physical units of each of the model parameters, shown in the table below:

Table 7.3: Physical meaning and units of parameters for the SCARA robot

Variable	Symbol	Dimension
Mass of the first link	m_1	kg
Mass of the second link	m_2	kg
Length of first link	L_1	m
Length of second link	L_2	m
Distance from link 1 pivot to C.G. 1	L_{c1}	m
Distance from link 2 pivot to C.G. 2	L_{c2}	m
Gravitational constant	g	$\mathrm{m/s^2}$
Moment of inertia, Link 1	I_1	$\mathrm{kg \cdot m^2}$
Moment of inertia, Link 2	I_2	$\mathrm{kg \cdot m^2}$
Torque on Link 1	τ_1	$\mathrm{kg \cdot m^2 \cdot s^{-2}}$
Torque on Link 2	τ_2	$\mathrm{kg \cdot m^2 \cdot s^{-2}}$
Time	t	s

Note that the states are angles, and hence are already dimensionless. The resulting dimensional matrix becomes, using g, m_1, L_1 as repeating parameters:

	m_2	L_2	L_{c1}	L_{c2}	I_1	I_2	τ_1	τ_2	t	m_1	L_1	g
kg	1	0	0	0	1	1	1	1	0	1	0	0
m	0	1	1	1	2	2	2	2	0	0	1	1
s	0	0	0	0	0	0	−2	−2	1	0	0	−2
π_1	1	0	⋯					⋯	0	−1	0	0
π_2	0	1	0						⋮	0	−1	0
π_3	⋮	0	1	0						0	−1	0
π_4			0	1	0					0	−1	0
π_5				0	1	0				−1	−2	0
π_6					0	1	0			−1	−2	0
π_7						0	1	0	⋮	−1	−1	−1
π_8	⋮						0	1	0	−1	−1	−1
π_9	0	⋯					⋯	0	1	0	−1/2	1/2

(7.12)

Which result in pi-values as:

$$\pi_1 = \frac{m_2}{m_2}, \pi_2 = \frac{L_2}{L_1}, \pi_3 = \frac{L_{c1}}{L_1}, \pi_4 = \frac{L_{c2}}{L_1}, \pi_5 = \frac{I_1}{m_1 \cdot L_1^2},$$
$$\pi_6 = \frac{I_2}{m_1 \cdot L_1^2}, \pi_7 = \frac{\tau_1}{m_1 \cdot L_1 \cdot g}, \pi_8 = \frac{\tau_2}{m_1 \cdot L_1 \cdot g}, \pi_9 = \frac{t \cdot g^{1/2}}{L_1^{1/2}} \quad (7.13)$$

Let us now consider two robot arms that are physically different in size and mass, but are dimensionally similar in the sense of the Pi-equivalence definition presented in Chapter 4. The parameters for the two arbitrary robots are given below.

Table 7.4: Parameters for two physically different robots that are dynamically similar

Variable	Symbol	Dimension	System 1	System 2
Mass of the first link	m_1	kg	1	2
Mass of the second link	m_2	kg	1	2
Length of first link	L_1	m	2	2
Length of second link	L_2	m	2	2
Distance from link 1 pivot to C.G. 1	L_{c1}	m	1	1
Distance from link 2 pivot to C.G. 2	L_{c2}	m	1	1
Gravitational constant	g	m/s^2	9.81	9.81
Moment of inertia, Link 1	I_1	$kg \cdot m^2$	1	2
Moment of inertia, Link 2	I_2	$kg \cdot m^2$	1	2

Let us assume that some controller, designed by whatever means, creates a desired motion on the first arm. For this simulation, we use a Kp of 100 and Kd of 10. To maintain similarity in the second arm, the seventh and eight pi-parameters require that the torque be exactly twice as high; which necessarily implies the gains should be twice as large. However, we note that the nature of the feedback linearization causes the resulting system to be parameterless. The torque increase needed is exactly balanced by the feedback linearization. The plots below demonstrate the controller on both systems. The first system is shown in bold blue, the second in red, and the reference command is in red. The reference command for both arms are pi/4-amplitude square waves, but the period of the second link is half that of the first link.

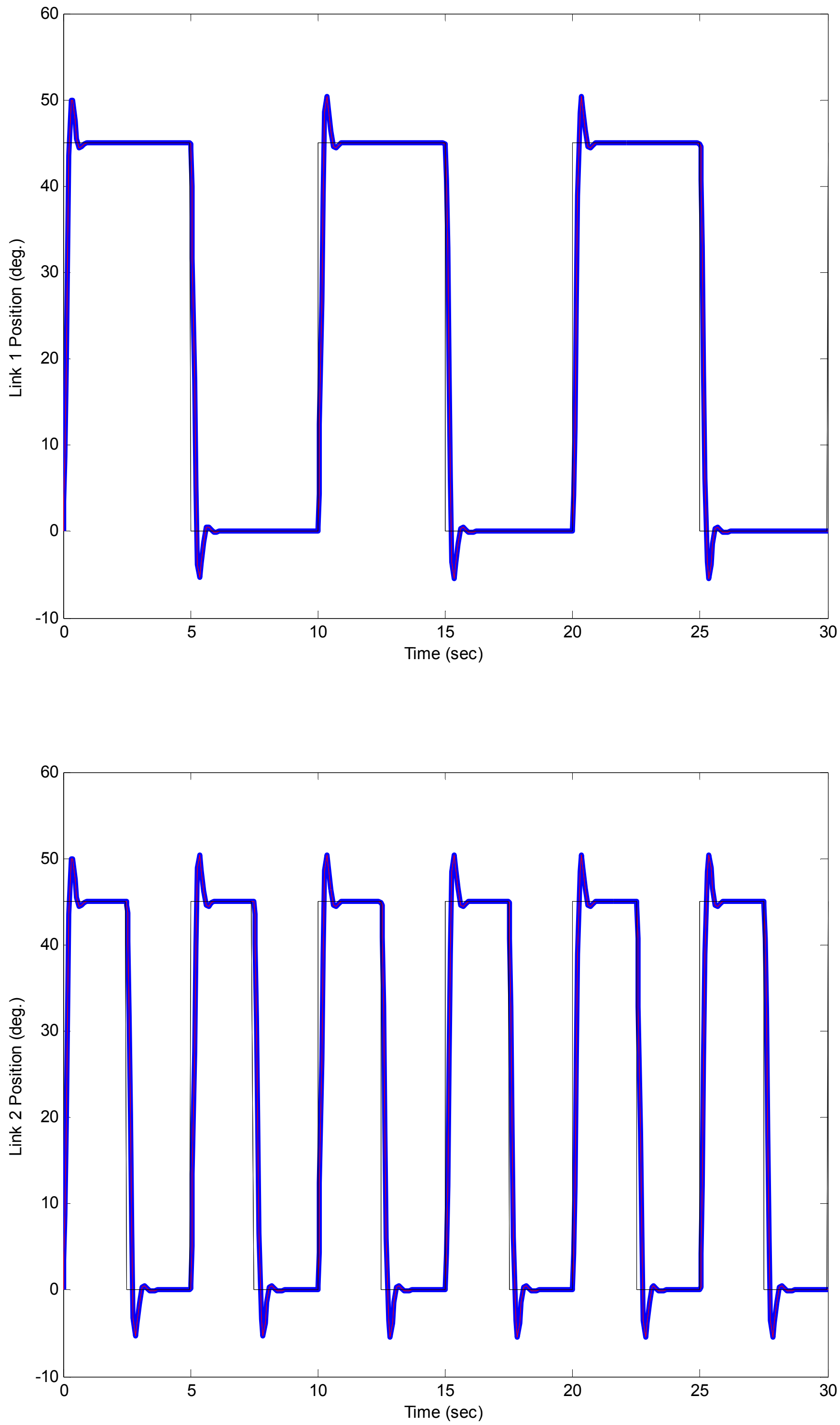


Figure 7.6: PD-computed torque implementation on both arms.

Clearly, the results on both arms are identical as expected. However, if we perturb any of the parameters, we find that the plots will always be identical because the feedback linearization is exact. The more interesting (and realistic) case occurs when the system is perturbed such that the feedback system is using parameters that do not match with the actual parameters. For instance, imagine that we perturb both system's design mass, m_1, to be 95% less than the actual system mass 1. This factor is large because the controller is very robust and also to emphasize the difference in performance. The following plots show the degradation in performance for the two systems:

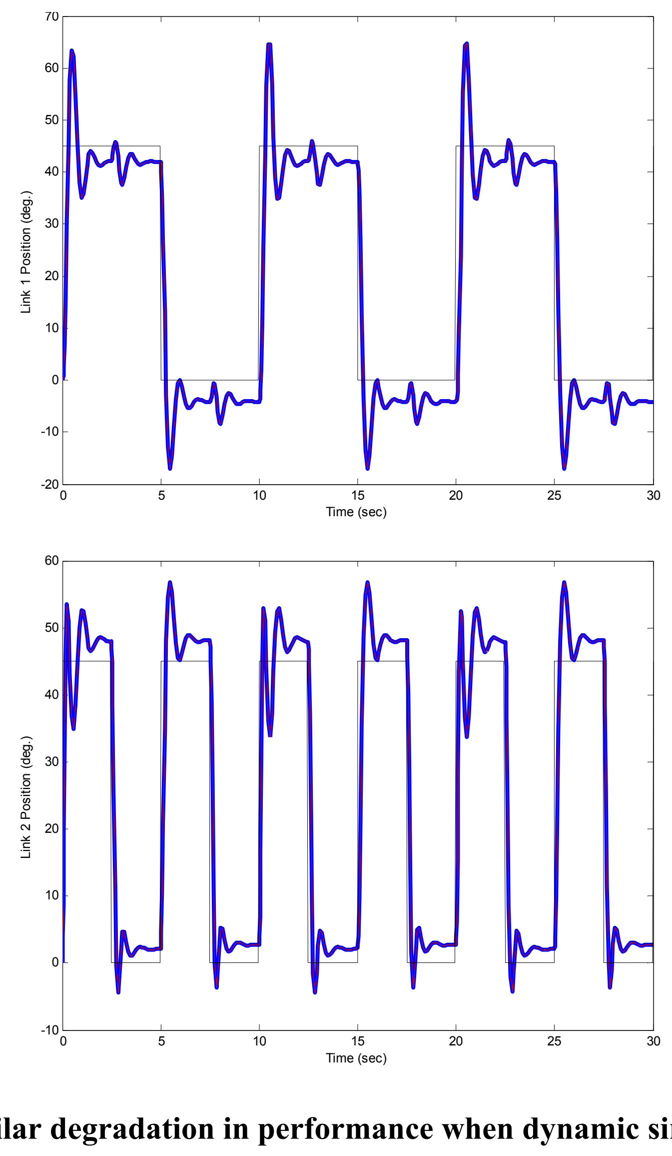


Figure 7.7: Similar degradation in performance when dynamic similarity maintained

Again, the plots are identical, but this is because dynamic similarity is maintained. However, if we change the second systems mass 1 link from a value of 2 kg to a value of 4 kg (which is a small change compare to the factor of 50 change considered earlier), dynamic similitude is violated. The system responses are clearly different, illustrated below:

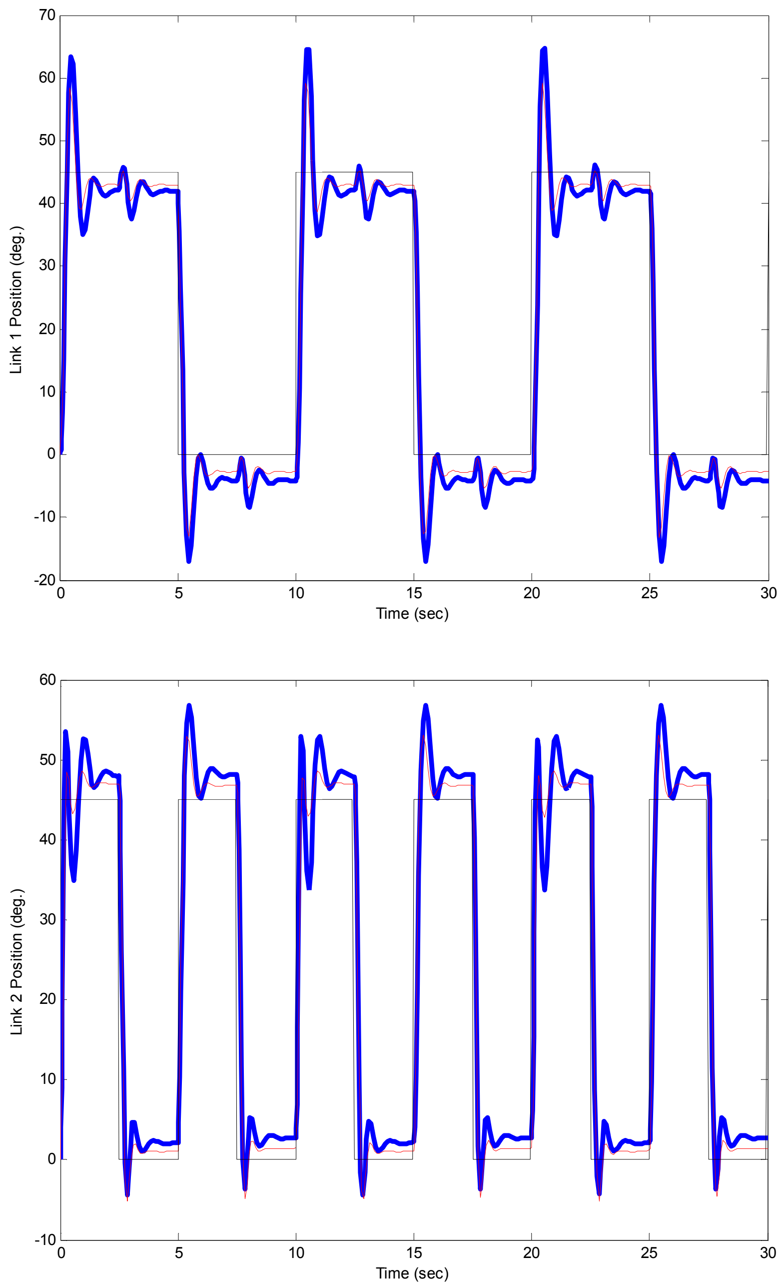


Figure 7.8: Dissimilar responses when dynamic similarity is not maintained

While sensitivity equations are not directly considered, it should be clear from previous chapters that a type of sensitivity invariance is clearly present. This invariance obviously allows an improved and generalized consideration of system robustness that is not evident by simple examination of the system parameters.

7.5 Future Work Related to Dimensional Analysis

7.5.1 Cascaded Systems and Dimensionless Reducibility

Given an arbitrary block diagram as in the figure below, one first notes that the diagram itself imposes dimensional constraints on the system signals and subsystems. However, many different control structures have equivalent closed-loop representations. A future research effort could be to investigate whether some representations have dimensional advantages in representation. Specifically, one can apply dimensional analysis to a control system at many levels. In the thesis, the analysis was generally applied for the entire loop, yet one might create dimensionless representations of each of the subsystems within the loop (the plant, the controller, the sensor), each with their own pi-values. At some point the benefits of additional dimensional constraints of 'unmasking' each subsystem will be outweighed by the increased number of pi-parameters needed for each subsystem. It is unclear at which level of model complexity that this tradeoff will transition from increasing benefit to increasing detriment to a control engineer.

What is specifically needed is some type of Theorem of Dimensionally Decoupled Subsystems, where one can state that a certain component-level modeling accuracy is required by dimensional analysis, and below this level there is no additional uniqueness in the model representation. Additionally, some investigation should explore dimensional rules of cascaded dimensionless systems.

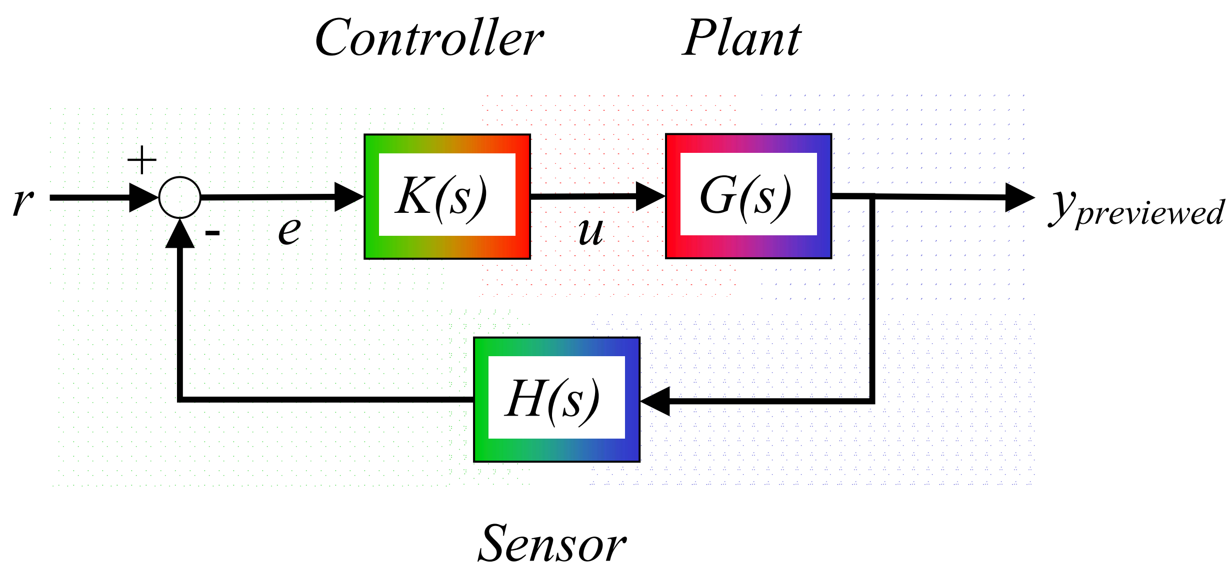


Figure 7.9: A generalized control loop has inherent dimensional constraints

7.5.2 Stability of Dimensional Unit Systems

A simple imaginative example is presented that illustrates the notion of the stability of a unit system. Before the definition of the meter was redefined based on a time unit (i.e. the length a photon will travel in a given time), the meter was defined literally by the horizontal distance spanned by a physical rod. Let us assume for a conceptual experiment that the meter is supported by a table of specified width, for instance 0.75 meters in width so that the ends of the rod stick out (which they did in practice to assist in measuring length). Now let us imagine that the material of the meter stick was chosen poorly (like a lead pipe), so that the rod relaxed in length over time. If one were making measurements based on the new meter stick, the table would, over time, appear to be measured slightly wider than specification, perhaps 0.753 meters. This is a result of the meter-stick standard shortening somewhat due to sagging. Let us assume that the table support is then shortened to meet specification, and therefore the meter stick sagged more. It isn't difficult to see that the dimensional unit system is changing. The stability of the change (whether it converges to a finite value or not) depends on the flexibility of the rod.

Obvious real-world situations indeed arise where dimensional systems are in continuous change, or worse, become unstable. Any professor familiar with grade inflation or any economist familiar with monetary inflation can describe first-hand the effects of unit instability.

In a control theoretic context, it would be very useful to be able to analyze control loops (like our economy) in situations where the unit system itself is in transition. Indeed, an interesting control problem would be to determine conditions for stability of a fixed control loop acting in the presence of a time-varying unit system. It is likely such a problem may be addressed in a dimensionless form (and perhaps already has). This conjecture remains for future consideration.

7.6 References

1. Bond, W. N. Phil. Mag. 7.94 (1929): 719-21.
2. Eslami, Mansour. Theory of Sensitivity in Dynamic Systems, An Introduction. New York: Springer-Verlag, 1994.
3. Sastry, Shankar. Nonlinear Systems: Analysis, Stability, and Control. New York: Springer-Verlag, 1999.
4. Slotine, Jean-Jaques E., and Weiping Li. Applied Nonlinear Control. Englewood Cliffs, NJ: Prentice Hall, 1991.
5. Spong, Mark W., and M. Vidyasagar. Robot Dynamics and Control. New York: John Wiley & Sons, 1989.

Appendix A: Dimensional Systems and Conversions

A.1 Scaling Prefixes

The following chart lists definitions of the most common scaling prefixes (Source: [1])

Power of 10 Multiplier	Prefix		Notes
	Name	Symbol	
18	Exa	E	
15	Peta	P	
12	Tera	T	
9	Giga	G	
6	Mega	M	
3	kilo	k	
2	hecto	h	no longer preferred
1	deca	da	no longer preferred
-1	deci	da	no longer preferred
-2	centi	c	use not suggested
-3	milli	m	
-6	micro	μ	
-9	nano	n	
-12	pico	p	
-15	femto	f	
-18	atto	a	

A.2 The Four Primary Dimensional Systems

While innumerable dimensional systems have been used (see for additional systems), today four primary systems are in usage [Langhaar, 1951 #917], shown in the figure below.

They are separated into two types depending on whether force or mass is used to generate one of its fundamental dimensions. Each of the systems is discussed below:

	metric	**American/British**
force based	MKS Force System	American/British Force (Engineering) system
mass based	SI	American/British Mass system

A.2.1 The SI

The most dominant system of the four is by far the SI. The term 'SI' is an abbreviation for *Le Systeme Internaional d'Unites*, or translated: "International System of Units". It is currently the most popular for the following reasons:

- It is agreed upon internationally, and therefore facilitates international communication
- It is simple, logically precise, and decimal based. This facilitates learning and technical calculation.
- It has a small number of fundamental dimensions, and a large number of derived dimensions
- It produces derived dimensions that all have unit coefficients (this is a subtle point that is discussed further in [Szirtes, 1997 #918].

The benefits are obvious enough not to require further elaboration.

While SI was adopted officially in the United States in 1960, after the *11th General Conference of Weights and Measures*, metric dimensions were made legal in the U.S. in 1866. The origin of the system dates back to 1791, when the French adopted the meter as a dimension of length, defined as 1/40,000,000 the circumference of the Earth as measured using a meridian passing through Paris. Very soon afterward, Lavoisier defined the metric mass by taking 1 cm^3 of distilled water at its maximum density (at 4 ^oC) and defined this as a *gramme* [Szirtes, 1997 #918]. In 1801, France begun the first large-scale attempt at a measurement standard with the introduction of what is now used as the metric system [Taylor, 1974 #935].

Since this early history, the definition of fundamental dimensions has changed as measurement accuracy has improved. A meter is now defined as the distance light travels in a vacuum in 1/299,792,458 of a second. The kilogram is now defined as the mass of a platinum-iridium cylinder kept at the *International Bureau of Weights and Measures* in Sevres, France.

The second is defined today as the duration of 9,192,631,770 periods of the radiation corresponding to a hyperfine level of the emission of the ground state of cesium-133 atoms.

The increasing temporal resolution of modern electronics will soon require a new definition of what are today considered fundamental units. This arises because modern computers are already operating in the range of 1,000,000,000 cycles per second. At this rate, it is already exceedingly difficult to synchronize information between two different circuits or sections of the same circuit that are physically separated (Sean reference Scientific American). If two separated circuits operating at 1 GHz (say two telescopes on separate sides of the earth) obtain one unit of data on each cycle for a second, yet one is 1 cycle faster than the other, the question of 'synchronizing' the two measurements is problematic but still feasible based on the atomic clock timing of a standard second. In the case of a 10 GHz computers or faster, the resolution of the computer is *finer* than the definitional resolution of the second, and the definition of time *between* each tick of the atomic clock becomes very troublesome and arbitrary.

In addition to these three units, there four other fundamental units in SI listed in the table below. Their exact definitions can be found in [Szirtes, 1997 #918].

Quantity	Dimension	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	Kelvin	K
amount of substance	mole	mol
luminosity	candela	cd

Figure A.1: The seven fundamental dimensions in SI

The most common of the derived units associated with SI are listed in the Appendix.

In addition to the SI system, an additional metric, mass-based system is the CGS system. The acronym CGS stands for *centimeter-gram-second*. The unit of force in CGS is the dyne, or the force necessary to impart 1 cm/s^2 acceleration to a mass of one gram. Thus, the dimension of force is $cm \cdot g \cdot s^{-2}$.

A.2.2 Metric, Force-based Systems

Known commonly as the MKS system, this system's dimensions are based on the *meter* (m), *kilogram* (kg), and *second* (s) as fundamental dimensions. In this usage, kilograms are a measure of *force*. As a consequence, mass is a derived dimension, with units of force per acceleration. Therefore:

$$[mass] = \frac{\text{kg}}{\left(\frac{\text{m}}{\text{s}^2}\right)} = \text{m}^{-1} \cdot \text{kg} \cdot \text{s}^2 \quad (1.1)$$

The usage of kilograms is sometimes written as (kgf) to distinguish it from the mass unit of (kg) in SI. In the MKS system, mass has no named unit, and is only designated by its dimensions. This unit system is used primarily in non-English speaking European countries, and is still favored in some areas [Szirtes, 1997 #918].

A.2.3 American/British Force-based (Engineering) System

In this system, the fundamental dimensions are the *foot* (ft), the *pound* (lb), and the *second* (s). Here, lb is a force measurement. Therefore, the dimension of mass is again defined by a force per acceleration:

$$[mass] = \frac{\text{lb}}{\left(\frac{\text{ft}}{\text{s}^2}\right)} = \text{ft}^{-1} \cdot \text{lb} \cdot \text{s}^2 \quad (1.2)$$

The unit of mass is called a *slug*, and is defined so that:

$$1\ \text{ft}^{-1} \cdot \text{lb} \cdot \text{s}^2 = 12\ \text{slugs} \quad (1.3)$$

Again, the *force* dimension lb is often written as lbf (pound-force) to distinguish it from the mass dimension lb in the mass-based system below.

A.2.4 American/British Mass-based (Scientific) System

In this system the *pound* (lb) is the fundamental unit of *mass*, and the *foot* (ft) and *second* (s) make the remaining fundamental units. In this unit system, the pound is established as 0.45359237 kg. The units of force are therefore derived from *mass* multiplied by acceleration.

$$[force] = \text{lb} \cdot \left(\frac{\text{ft}}{\text{s}^2}\right) = \text{ft} \cdot \text{lb} \cdot \text{s}^{-2} = 1 \text{ poundal} \qquad (1.4)$$

This unit system is sometimes referred to as the *Imperial Scientific System*.

A.3 Common SI Units

A.3.1 Fundamental and Named Derived Units

The most common SI units are listed below, presented from tables developed in [1], where detailed descriptions of each term can be found. Fundamental units are listed in gray.

Quantity	SI Dimension	Typical Form	Name	Symbol
length	m	m	meter	m
mass	kg	kg	kilogram	kg
time	s	s	second	s
electric current	A	A	ampere	A
temperature	K	K	kelvin	K
amount of substance	mol	mol	mole	mol
luminosity	cd	cd	candela	cd
radioactivity	s^{-1}	s^{-1}	becquerel	Bq
quantity of charge	$s \cdot A$	$s \cdot A$	coulomb	C
Celcius temperature	K	K	degree Celcius	^{o}C
capacitance	$m^{-2} \cdot kg^{-1} \cdot s^{4} \cdot A^{2}$	C/V	farad	F
dose of radiation	$m^{2} \cdot s^{-2}$	J/kg	gray	Gy
inductance	$m^{2} \cdot kg \cdot s^{-2} \cdot A^{-2}$	Wb/A	henry	H
frequency	s^{-1}	s^{-1}	hertz	Hz
energy, work, heat quantity	$m^{2} \cdot kg \cdot s^{-2}$	$N \cdot m$	joule	J
luminous flux	cd	$cd \cdot sr$	lumen	lm
illuminance	$m^{-2} \cdot cd$	$lm \cdot m^{2}$	lux	lx
force	$m \cdot kg \cdot s^{-2}$	$m \cdot kg \cdot s^{-2}$	newton	N
electric resistance	$m^{2} \cdot kg \cdot s^{-3} \cdot A^{-2}$	V/A	ohm	Ω
pressure, stress	$m^{-1} \cdot kg \cdot s^{-2}$	$N \cdot m^{-2}$	pascal	Pa
electric conductance	$m^{-2} \cdot kg^{-1} \cdot s^{3} \cdot A^{2}$	A/V	siemens	S
dose equivalent of ionizing radiation	$m^{2} \cdot s^{-2}$	J/kg	sievert	Sv
magnetic flux density	$m^{-2} \cdot kg^{-1} \cdot s^{4} \cdot A^{2}$	$Wb \cdot m^{-2}$	tesla	T
electric potential, potential difference	$m^{2} \cdot kg \cdot s^{-3} \cdot A^{-1}$	V	volt	V
power, radiant flux	$m^{2} \cdot kg \cdot s^{-3}$	J/s	watt	W
magnetic flux	$m^{2} \cdot kg \cdot s^{-2} \cdot A^{-1}$	$V \cdot s$	weber	Wb

A.3.2 SI Units Related to Mechanics and Heat

Quantity	SI Dimension
area	m^2
volume	m^3
density	$m^{-3} \cdot kg$
weight	$m \cdot kg \cdot s^{-2}$
heat transfer coefficient	$kg \cdot s^{-3} \cdot K^{-1}$
moment of force	$m^2 \cdot kg \cdot s^{-2}$
linear velocity	$m \cdot s^{-1}$
angular velocity	s^{-1}
linear acceleration	$m \cdot s^{-2}$
angular acceleration	s^{-2}
linear jerk	$m \cdot s^{-3}$
moment of inertia	$m^2 \cdot kg$
gravitational acceleration	$m \cdot s^{-2}$
dynamic viscosity	$m^{-1} \cdot kg \cdot s^{-1}$
kinematic viscosity	$m^2 \cdot s^{-1}$
impulse	$m \cdot kg \cdot s^{-1}$
moment of momentum	$m^2 \cdot kg \cdot s^{-1}$
specific heat capacity	$m^2 \cdot s^{-2} \cdot K^{-1}$
heat capacity	$m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}$
specific entropy	$m^2 \cdot s^{-2} \cdot K^{-1}$
entropy	$m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}$
universal gravitational constant	$m^3 \cdot kg^{-1} \cdot s^{-2}$
thermal conductivity	$m \cdot kg \cdot s^{-3} \cdot K^{-1}$
thermal resistivity	$m^{-1} \cdot kg^{-1} \cdot s^3 \cdot K^1$
specific volume	$m^3 \cdot kg^{-1}$
energy density	$m^{-1} \cdot kg \cdot s^{-2}$
surface tension	$kg \cdot s^{-2}$
wave number	m^{-1}
wave length	m
momentum	$m \cdot kg \cdot s^{-1}$
second moment of area	m^4
stress (normal, shear)	$m^{-1} \cdot kg \cdot s^{-2}$
heat flux	$m^2 \cdot kg \cdot s^{-3}$
angular jerk	s^{-3}
Young's modulus	$m^{-1} \cdot kg \cdot s^{-2}$
modulus of shear	$m^{-1} \cdot kg \cdot s^{-2}$
compressibility	$m \cdot kg^{-1} \cdot s^2$
flow rate (mass)	$kg \cdot s^{-1}$
flow rate (volume)	$m^3 \cdot s^{-1}$
specific energy	$m^2 \cdot s^{-2}$
linear expansion coefficient	K^{-1}
enthalpy	$m^2 \cdot kg \cdot s^{-2}$
specific enthalpy	$m^2 \cdot s^{-2}$
volumetric expansion coefficient	K^{-1}
linear density	$m^{-1} \cdot kg$
area density	$m^{-2} \cdot kg$
material permeance	$m^{-1} \cdot s$

A.4 Standard Dimensional Conversions

A.4.1 Length Conversions

1 meter = 39.37008 inches = 3.28084 feet = 1.09361 yards

1 meter = 6.21371E-4 miles = 5.39957E-4 nautical miles

1 meter = 6.68459E-12 astronomical units = 1.05702E-16 light-years = 3.24078E-17 parsecs

A.4.2 Volume Conversions

1 liter = 0.001 cubic meters = 0.21997 UK gallons = 0.26417 US gallons

1 liter = 2.11338 US pints = 1.05669 US quarts

A.4.3 Mass Conversions

1 kilogram = 0.10197 $kgf \cdot s^2/m$ = 2.20462 pounds = 6.85218E-2 slugs = 5.71015E-3 $lbf \cdot s^2/in$

A.4.4 Force Conversions

1 Newton = 1E5 dynes = 0.22481 pounds-force = 7.23301 poundals = 0.10197 kilograms-force

A.4.5 Energy Conversions

1 joule = 9.47817E-4 BTU = 0.23885 calories = 2.77777E-7 kW-hours

1 joule = 0.73756 lbf·ft = 0.10197 kgf·m = 1.11265E-17 kg (relativistic mass/energy)

A.4.6 Power Conversions

1 watt = 1.35962E-3 HP (metric) = 1.34102E-3 HP

A.4.7 Pressure Conversions

1 pascal = 9.86923E-6 atmospheres = 1.45038E-4 psi = 7.50062E-3 torr

A.4.8 Magnetic Flux

1 tesla = 1E4 gauss = 1E9 gammas = 1E4 $maxwells/cm^2$

A.4.9 Temperature Conversions

Kelvin to Celsius: $^0C = K - 273.15$

Kelvin to Fahrenheit $^0F = 9*K/5 – 459.67$

Kelvin to Rankine $^0R = 9*K/5$

A.5 Lexicographical Rules for Dimensions

The following generally accepted lexicographical rules were adopted from [1].

1. Do not abbreviate text to express a dimension. For example 30 cubic centimeters is 30 cm^3, not 30 cc; 40 square meters is 40 m^2, not 40 sq. m.
2. Write (print) all SI symbols in upright (roman) type, irrespective of the typeface used in the surrounding text. Upright excludes *italic* and other sloped typefaces. For example, *The car is 2.4* m *long*, not *The car is 2.4 m long.*
3. Observe the lower and uppercase letters for symbols. In general, symbols should be written in lowercase letters, except named SI units, which when *abbreviated*, are written in capital letters. For example, *The current is 4 ampere*, or *The current is 4* A, not *The current is 4 Ampere*. If the abbreviation of a named derived SI unit is composed of 2 characters, then the first character is uppercase and the second lowercase. For instance, *5 weber = 5* Wb.
4. Do not affix an "s" to any *symbol* of dimension to indicate plurality. For example, use 19 kg, not 19 kgs, but the usage *19 kilograms* is correct, since 'kilograms' is not a symbol.
5. Do not put a period at the end of an abbreviation of a dimension, except if this abbreviation is at the end of the sentence. For instance, "*The table is 20* m *long*," not "*The table is 20 m. long*." However, "*The length of the table is 20 m.*" is correct.
6. Put a space between the last digit of a magnitude and its dimension, whether the dimension is abbreviated or not. For example, "*The mass is 20* kg.*"*, not *"The mass is 20*kg". The only exception is when writing degrees Celcius, when a space should not appear between the magnitude and the degree symbol, for instance 20^oC, not 20 ^oC.
7. Do not begin a sentence with a symbol or dimension.
8. Use lowercase letters for unabbreviated named derived SI units. For instance, *"The force was 23 newtons,"* not *"The force was 20 Newtons"*. The only exception is that the word Celsius is always capitalized.

9. Do not mix names and symbols in a dimension. For instance, 20 m/s, not 20 meters/s or 20 m/second.
10. Do not attach a quantifier to a dimension. If necessary, attach the quantifier to the magnitude of the unit in question. For example, *"The gage pressure is 250* psi*"*, not *"The pressure is 250* psig*"*.
11. Do not put a space between the prefix and the symbol (or name) of a dimension. For example, 8 km, not 8 k m. The prefix and the SI symbol form a new symbol, and should be treated as such.
12. Do not compound prefixes, i.e. multiple prefixes are not allowed. For example, a 3 Mt explosion, not a 3 kkt explosion.
13. Do not use more than one unit (base, multiple, or submultiple) to describe a quantity. For example, L = 3.218 m, not L = 3 m 23 cm 8 mm. Exception, phase angles and time units.
14. Do not use more than one prefix in a dimension, and if possible apply this single prefix in the numerator of the dimension. For instance, 0.003 m/s may be written as 3 mm/s, not 3 km/Ms or 3 m/ks.
15. To avoid ambiguity, always place a dot between two units (dimensions) to indicate multiplication. For example, m·N means (meter) x (newton), but mN means millinewton.
16. Do not use the solidus (/) symbol more than once in any dimensional expression, unless parenthesis are used. For example, use m/s^2 not m/s/s.
17. Do not substitute the dimension of a quantity for its name. For example, *"The area of the property is 23* acres*"*, not *"The acreage is 23"*.
18. When the name of a dimension appears in text and a division is indicated, use the word "per" and not the solidus symbol (/). For example, it is not *8 newton/square meter* but *8 newton per square meter*. However, 8 N/m^2 is correct.

A.6 References

1. T. Szirtes, *Applied Dimensional Analysis and Modeling*, 1st ed. New York: McGraw-Hill, 1997.

Appendix B: Dimensional Publications and Examples

A large number of examples have been presented in these publications on dimensional analysis, and they are often repeated with only slight variations. Since it is often insightful to seek out a particular example, a listing of such primary examples was created and given below. While this list is not all inclusive, it certainly illustrates the breadth of the topic.

B.1 Dimensional Publications

P.W. Bridgman's book, written in 1920 and published in 1922, is considered the first text dedicated to the topic of dimensional analysis. Bridgman was a professor of Physics at Harvard University and was an associate of Dr. Buckingham and M. Hershey, both at the Bureau of Standards. As a first publication, it was strongly influential in establishing dimensional methods of analysis from a parameter-based viewpoint. While this approach is the most useful for arbitrary problems, it has 'flavored' the field in that the subject is generally taught and understood to be solely a parameter-based approach. It is this persistent misconception that has prevented many dimensional techniques from entering into the mainstream considerations of modern control theory.

Bridgman's contemporary and sometimes critic, Campbell, published two books on the philosophy of measurement, *Physics; The Elements* in 1920 and *Account of the Principles of Measurement and Calculation* in 1928. Campbell's contribution to the field was a strong consideration as to the nature and meaning of a physical measurement, and hence dimension.

Langhaar, as a professor of Theoretical and Applied Mechanics at the University of Illinois, taught dimensional analysis at a graduate level and was urged to publish by his colleague, Prof. B.B. Seely. His book, written in 1951, presents a view of dimensional analysis that is one of the first explicitly based on the concept of a dimensional basis. This concept is not yet fully developed, yet the concept of a dimensional matrix of basis vectors and the corresponding notions of dimensional rank are fully seen in this work. His dimensional algebraic approach serves as the model for most modern publications on the subject, and his book should be one of the first references for a control theorist interested in the subject.

H. E. Huntley, Professor of Physics at University College of the Gold Coast, published his book Dimensional Analysis in 1952. The book is written with an undergraduate focus, and is relatively light reading both in depth and in ease of understanding. However, it is interesting that in Huntley's view, the concept of fundamental versus derived dimensions was still not resolved (authors of his time, Langhaar and others, showed that the number of fundamental dimensions is arbitrary). Huntley (correctly) infers that the choice of fundamental dimension should be decided by the problem at hand.

C. M. Focken, Director of the Museum of Applied Science of Victoria, an Reader in Physics in the University of Otago, presented his book in 1953 which basically served as a restatement of Bridgman's work with additional discussion of Campbell's critiques of Bridgman's and Buckingham's approach. Focken was a contemporary of Herbert Dingle, who published on the topic of Dimensional Analysis. In particular, a significant amount of focus is given to the electromagnetic dimensional systems. However, it is clear that Focken has not expressed in his writing an understanding of the dimensional basis approach of Langhaar, although at the time of writing had begun to study Langhaar's method.

Also in 1953, W. J. Duncan published his book, *Physical Similarity and Dimensional Analysis: An Elementary Treatise*. Dr. Duncan was the Mechanics Professor of Aeronautics and Fluid Mechanics at the University of Glasgow, Briton. The unique aspect of his work is that it is one of the first to discuss dimensional analysis with the focus on physical similarity between disparate systems. By this nature alone, it is one of the better texts to read for an engineering student first learning the concepts of dimensional analysis. Additionally, it presents one of the best proofs of the Buckingham Pi Theorem, discussed later.

Between 1943 and 1961, L.I. Sedov of the Academy of Sciences, U.S.S.R. published four editions of his pivotal work on dimensional analysis specifically focusing on applications to differential equations, *Similarity and Dimensional Methods in Mechanics*.

In 1965, Stephen J. Kline published his book, [Kline, 1965 #936], *Similitude and Approximation Theory*, and at the time he was a Professor of Mechanical Engineering at Stanford University. This book presents several methods of dimensional analysis, with a strong (and healthy) focus on differential and integral equations built off of Sedov's work. Within Kline's view, there are several approaches to dimensional analysis and the approach of the Buckingham Pi theorem is only one of three. The remaining two are proportionality of forces (which he calls *fractional analysis*) and homogeneity of the governing equations (which will be discussed in Chapter 4 of this thesis). Kline provides a critique of the traditional approach, but many of these criticisms are easily addressed in this thesis by taking a *dimensional basis* viewpoint. While his proposal of fractional analysis is certainly a more powerful technique, it requires significant amount of assumptions and thus a very *knowledgeable* user. For general controls usage, it is simpler to present the more traditional approach modified with well-known linear-algebraic checks for singularity checks and basis transformations.

Ellis, writing in 1966 in his book *Basic Concept of Measurement*, claims that no significant work in the philosophical consideration of measure met had been presented since Bridgman. Dr. Ellis, a senior lecturer in History and Philosophy of Science at the University of Melbourne, was a student of Prof. J. J. C. Smart, who taught him concepts of dimensional analysis from the works of Mach and Campbell. Ellis' criticism of the modern concept of measurment is simply a more formalized restatement of O'Rahilly's definition of dimensions, but his book serves as a basic refresher on the meaning, philosophy, and use of measurement. However, Ellis' work is very dry and should be reserved for the true insomniac.

B.S. Massey, a Reader in Mechanical Engineering at the University College in London, wrote his book *Units, Dimensional Analysis and Physical Similarity* in 1971. This book is primarily a compendium of unit definitions, unit systems, and definition of dimensionless pi-parameters in common use in science and engineering. The sections on dimensional analysis present no new examples that were not already well developed by others.

In 1974, E. S. Taylor, Professor Emeritus from the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology, published his book titled *Dimensional Analysis for Engineers* [Taylor, 1974 #935]. This book presents the material from a decidedly Mechanical Engineering standpoint. It is notable for the focus on usage of dimensional analysis on problems where dynamics are only partially understood. Several unique project applications are presented at the conclusion of the book that by themselves are worth reading for any mechanical engineer.

By 1975, the notion is firmly established that basis (or fundamental) dimensions are largely a matter of convention. Whereas previous authors (notably Ellis) dedicate great consideration to this argument of dimensioning as a *relative* versus absolute property, this is simply accepted by Isaacson from the start, writing the book *Dimensional Methods in Engineering and Physics* in 1975. At this point, matrix representations and solutions of dimensional basis become common (see section 4.2 of this reference).

The most modern entry in the many publications on dimensional analysis is the book by Thomas Szirtes. His book, *Applied Dimensional Analysis and Modeling* [2], cites Langhaar as instrumental in his work both at NASA and RCA. Szirtes book on its own is a very detailed (790 pages!) and example-oriented overview of the present field of dimensional analysis as applied to parameter-based, dimensional approaches. It serves as an excellent source of examples and references to the subject.

<u>History</u>:

1922	*P.W. Bridgman	Dimensional Analysis
1943	L. I. Sedov	Similarity and Dimensional Methods in Mechanics
1951	*Henry L. Langhaar	Dimensional Analysis and Theory of Models
1952	H.E. Huntley	Dimensional Analysis
1953	W. J. Duncan	Physical Similarity and Dimensional Analysis: An Elementary Treatise
1953	C. M. Focken	Dimensional Methods and Their Applications
1965	Stephen J. Kline	Similitude and Approximation Theory
1966	Brian Ellis	Basic Concepts of Measurement
1971	B.S. Massey	Units, Dimensional Analysis and Physical Similarity

1974	E.S. Taylor	Dimensional Analysis for Engineers
1975	Isaacson and Isaacson	Dimensional Methods in Engineering and Physics

B.2 Physics

B.2.1 Physical Mechanics

Period of a Swinging Pendulum	Bridgman, 1922	1-3, 82
	Huntley, 1952	23-5
	Focken, 1953	151 (prob. 1)
	Duncan, 1953	41-4, 117
	Kline, 1965	31-2
	Massey, 1971	56-7, 74
	Taylor, 1974	48-9
	Isaacson, 1975	27-29, 143-4
(Exact solution)	Duncan, 1953	33-4
Period of Compound Pendulums	Huntley, 1952	25-6
	Taylor, 1974	79-81
Period of a Conical Pendulum	Huntley, 1952	88-9
Distance/Time Relation for a Falling Object	Huntley, 1952	18-21
Range of a Projectile	Huntley, 1952	72-3, 77-8
	Duncan, 1953	32
Maximum Acceleration of a Uniform Elastic Sphere on Impact with a Wall Perpendicular to Velocity		
	Taylor, 1974	42-4
Stress on Wall During Sphere Impact (above)	Taylor, 1974	44
Differential Equation of Simple Harmonic Motion	Isaacson, 1975	38
Period of a Swinging of an Elastic Pendulum	Bridgman, 1922	59-65
Centripetal Force at Uniform Velocity	Focken, 1953	151 (prob. 2)
Angular Acceleration of a Disk	Focken, 1953	151 (prob. 3)
Acceleration of a Point Moving at Uniform Circular Motion		

Huntley, 1952 48

Tension in a Unif. Circular Rod Rotating In It's Own Plane About an Axis Through Its Center

Huntley, 1952 50-1

Resolving Power of a Telescope by Size and Wavelength

Focken, 1953 152 (prob. 10)

Intensity of Light Scattering from Small Particles (Rayleigh Scattering – Why the Sky is Blue?)

Raleigh (ref. in Huntley, p 62)

Huntley, 1952 62-3

Focken, 1953 152 (prob. 11)

Photoelectric Effect Focken, 1953 152 (prob. 24)

Range of a Fired Projectile Isaacson, 1975 70-1

Angle Deformed by Expansion of Bimetallic Strip Isaacson, 1975 73-5

Expansion of the Universe Isaacson, 1975 200-2

Dimensionless Form of Lagrange Dynamics (Froude Number)

Duncan, 1953 59-60

Attraction of Any Point Inside a Uniform Hollow Sphere by an Inverse-Square Force Law

Duncan, 1953 149

B.2.2 Electricity and Magnetism

Wavelength of Emission Spectra from a Solid (Einstein's Example and Debye's Proof)

Bridgman, 1922 89

Isaacson, 1975 189

Unified Theory of Gravitation and Electromagnetism Bridgman, 1922 90-91

Isaacson, 1975 189

Critical Mass of Uranium Causing Fusion Focken, 1953 112

Deflection of a Charged Particle due to a Magnetic Field Focken, 1953 148

Deflection due to Magnetic and Electric Fields Focken, 1953 148

Brownian Motion of a Particle Isaacson, 1975 194-5

B.3 Engineering

B.3.1 Civil Engineering

Weight of Rock needed for a Breakwater	Langhaar, 1951	11 (p)
Weight of a Granite Monument	Focken, 1953	178
Twist of an Bar by a Moment	Isaacson, 1975	69
Twist of an Elastic Shaft from an Applied Torque	Taylor, 1974	19
Bending Moment of a Beam	Langhaar, 1951	27 (prob. 7)
Deflection (or Frequency) of a Loaded Bar (Cantilever)	Bridgman, 1922	67-69
	Langhaar, 1951	28 (prob 26)
	Duncan, 1953	58-9,141-5
	Kline, 1965	32-4, 118-123
(Generalized solution … very nice)		171-172
	Taylor, 1974	39
	Isaacson, 1975	81-82, 145
(with flange)	Kline, 1965	124-7
Natural Modes of Elastic Structures	Langhaar, 1951	27 (prob. 16)
Similarity Conditions for the Deflection of Nonlinear Elastic Structures		
	Taylor, 1974	70-1
Frequency of Forced Vibration of a Structure	Langhaar, 1951	96
	Taylor, 1974	77-8
Frequency of Vibration of a Bar in Torsion	Taylor, 1974	23-4, 39
Frequency of Vibration for a Tuning Fork	Huntley, 1952	67-8
Frequency of Vibration for Bells and Chimes (Similarity Conditions)		
	Taylor, 1974	78
Deformation of Truss Systems Under External Loads	Taylor, 1974	7-8, 20-1, 69
Similarity Conditions for Structures Under Gravity Loads (Bridges)		
	Taylor, 1974	72-3
PDE for Free Transverse Vibrations in a Bar (Rayleigh)	Isaacson, 1975	39
Amplitude of Oscillation of Structures due to Wind	Langhaar, 1951	45 (prob. 22)

Frequency of Vibration of a Wire due to Wind	Langhaar, 1951	27 (prob. 22)
Frequency of Vibration of a Stretched Elastic String or Wire		
	Huntley, 1952	66-7
	Focken, 1953	14, 127
Energy of a Vibrating Wire (Fundamental Mode)	Huntley, 1952	66-7, 86
	Isaacson, 1975	49-51, 71-2
Frequency of Vibration of Fixed and Unfixed Wires	Focken, 1953	152 (prob. 12)
Relationship between Tension and Sag in a Stretched, Horizontal Wire		
	Huntley, 1952	42, 49-50
Vertical Wire with Hanging Weight is Twisted (Measuring Modulus of Rigidity)		
	Huntley, 1952	90-2
Velocity of a Wave Motion Through a Wire	Langhaar, 1951	27 (prob. 23)
	Huntley, 1952	64-5
Volume of Fluid Flow over a Spillway	Langhaar, 1951	28 (prob. 28)
Volume of Fluid Flow over a Weir	Langhaar, 1951	45 (prob. 19)
Stresses in an Arched Dam	Langhaar, 1951	92
Stresses in an Airport Runway Pavement	Langhaar, 1951	92-93
Stresses of Bridges	Langhaar, 1951	97 (prob. 1)
Buckling Stability of Loaded Columns	Duncan, 1953	120
Time-Dependent Run-Off from Illinois Watersheds	Langhaar, 1951	111

B.3.2 Fluid Mechanics

B.3.2.1 Theoretical Fluid Mechanics

Dimensional Considerations of Navier-Stokes Equations	Duncan, 1953	61-2
	Isaacson, 1975	108-112
Navier-Stokes Equations for Flow over an Immersed Object		
	Kline, 1965	127-37
Karman Similarity Criteria for Turbulent Shear Layers	Kline, 1965	137-143

Thermal-Conductive Boundary Layers (Variance of Temperature with Depth for Periodic Heating)

	Kline, 1965	143-151
Dimensional Considerations for Compressible but Invicid Flow		
	Duncan, 1953	74-8
Dimensional Considerations for Compressible and Viscous Flow		
	Duncan, 1953	78-80, 94
Similarity Rules for Supersonic and Subsonic Flow	Kline, 1965	172-181
Flow Near an Oscillating Flat Plate	Isaacson, 1975	112
Flow Near a Rotating Flat Disk	Isaacson, 1975	113-4
A Suddenly Accelerating Flat Plate	Kline, 1965	181-6
Steady Laminar Boundary Layer on a Flat Plate	Kline, 1965	186-189
Velocity Distribution Near a Flat Plate	Isaacson, 1975	117
Clapyron's Equation of State	Focken, 1953	114
Effect of Temperature on Viscosity of a Gas (Maxwell's Law of Gaseous Viscosity)		
	Rayleigh	?
	Langhaar, 1951	41
	Huntley, 1952	56-7
	Focken, 1953	138
Thermal Conductivity of a Gas	Isaacson, 1975	98-9
Thermal Conductivity of a Gas Based on Molecular Properties		
	Huntley, 1952	120-2
Repulsive Force Between Gas Molecules	Isaacson, 1975	99-100
Determining Molecular Forces from Temp/Viscosity Relationship		
	Huntley, 1952	119-20
Ideal Gas Law	Huntley, 1952	118-9
	Focken, 1953	137
Mean Free Path of Molecules of a Gas	Duncan, 1953	81
Speed of Sound in a Gas	Langhaar, 1951	17, 27 (prob)
	Huntley, 1952	65-6
	Focken, 1953	135
Speed of Sound in a Solid	Langhaar, 1951	27 (p)

Pressure Exerted by an Ideal Gas	Bridgman, 1922	70-2
Atmospheric Pressure as a Function of Height	Isaacson, 1975	22-3
Bernoulli's Theorem for Fluid Flow	Isaacson, 1975	23-4
Laminar, 2-Dimensional Jet	Kline, 1965	189-196

B.3.2.2 Applied Fluid Mechanics

Thrust Force of a Propeller	Langhaar, 1951	65
	Focken, 1953	52, 179-81
	Duncan, 1953	119
	Massey, 1971	66-68
	Isaacson, 1975	54-6
The Fan Laws (Axial Flow Fans and Propellers)	Duncan, 1953	125-33
	Isaacson, 1975	147-50
Power Produced by a Windmill	Duncan, 1953	132
Centrifugal and Axial Pumps	Langhaar, 1951	113, 115
	Duncan, 1953	133-4
	Kline, 1965	29-31
	Taylor, 1974	53-5
Effect of Viscosity on Efficiency of Centrifugal Pumps	Taylor, 1974	60-1
Pumps Handling Compressible Fluids: Similarity Conditions		
	Taylor, 1974	62-6
Speed of Cavitation of a Propeller	Langhaar, 1951	118 (prob. 17)
	Duncan, 1953	85-6
Specific Speed of a Hydraulic Turbine	Duncan, 1953	134-135
Run-Away Speed of Hydraulic Turbines	Langhaar, 1951	118 (prob. 9)
Power to Drive an Electric Fan	Taylor, 1974	40-2
Friction on the Wall of a Flume	Langhaar, 1951	7
Air Flow Through a Nozzle	Langhaar, 1951	12, 27 (p)
Measuring Flow of Incompressible Fluid by Means of a Sharp-Edged Orifice		
	Taylor, 1974	49-51
Volume of Fluid Through a Nozzle	Langhaar, 1951	28 (prob. 27)

Volume of Fluid Through a Tube	Huntley, 1952	54-6
Mass of Fluid Through a Tube	Huntley, 1952	82-3
Pressure Drop in a Uniform Pipe	Langhaar, 1951	22-24
	Massey, 1971	59-63, 65
Pressure Drop of Steady, Fully Established, Laminar, Incompressible Flow of Newtonian Fluid Through a Circular Pipe		
	Kline, 1965	18-20, 40-1, 62-3
Pressure Drop in a Smooth Pipe	Duncan, 1953	68
Poiseuille and Hagen Capillary Flow, Pressure/Flow Relation For Small Pipes		
	Duncan, 1953	70
	Taylor, 1974	12-3, 33-4
Flow Drop over an Orifice, Pipe Bend, or Obstruction	Langhaar, 1951	27 (p)
Flow of a Viscous Fluid Through a Small Pipe	Isaacson, 1975	82-6
Terminal Velocity of Spheres (laminar) (in Viscous Liquid) (Stoke's Problem) (Millikan drop)		
	Bridgman, 1922	65-67
	Huntley, 1952	95-8
	Focken, 1953	132-4
	Massey, 1971	69-71
	Taylor, 1974	14
Terminal Velocity of Spheres (turbulent)	Langhaar, 1951	45 (prob. 16)
Terminal Velocity of a Raindrop	Langhaar, 1951	45 (prob. 21)
	Huntley, 1952	57-8
	Isaacson, 1975	146-7
Drag of Smooth Spheres in an Incompressible Fluid	Langhaar, 1951	15-17, 19
Drag on an Immersed Body	Duncan, 1953	63
Drag on an Aircraft Wing	Langhaar, 1951	27 (p)
Lift and Drag on an Aircraft Wing	Duncan, 1953	64-5
Drag on a Ship	Langhaar, 1951	20-22, 40
	Duncan, 1953	95

Drag on a Ship (Scaling problems)	Isaacson, 1975	177-9
Skin Drag of a Ship	Langaar, 1951	118 (prob. 15)
Drag on a Sphere	Langhaar, 1951	66
Drag on an Arbitrary Shape	Bridgman, 1922	82-84
	Focken, 1953	183-187
Drag On a Body in a Viscous, Incompressible Fluid	Isaacson, 1975	151-3
Drag On a Sphere Moving Through Viscous Fluid	Taylor, 1974	51-2
Drag on a Flat Plate	Isaacson, 1975	119
Drag (Skin Friction) On an Elliptical Cylinder	Isaacson, 1975	119-21
Optimal Flight Speed of a Jet Aircraft	Duncan, 1953	135-9
Capillary Fluid Rise	Langhaar, 1951	27 (p)
	Huntley, 1952	58-9, 87
	Isaacson, 1975	76-8
Rise of a Lake Due to Steady Wind	Langhaar, 1951	27 (p)
Speed of Wind that Causes Ripples on Water	Langhaar, 1951	44 (prob. 12)
	Duncan, 1953	50-2
Speed of Wind that Causes White Caps on Water	Langhaar, 1951	44 (prob. 11)
Speed of Waves in Deep Water (Gravity Waves)	Rayleigh, 1915	Nature
	Bridgman, 1922	56-8
	Langhaar, 1951	27 (prob. 17)
	Huntley, 1952	63-4
	Focken, 1953	152 (prob. 5)
	Duncan, 1953	48-9
	Isaacson, 1975	43-4, 46-8
Speed of Waves in Shallow Water	Duncan, 1953	49-50
	Isaacson, 1975	45
Speed Waves by Surface Tension (Ripples)	Langhaar, 1951	45 (prob. 15)
	Huntley, 1952	61-2
	Focken, 1953	152 (prob. 5)
	Duncan, 1953	50-2

Energy of Water Waves Confined Between Two Plates	Huntley, 1952	93-5
Weight of a Drop of Water Dripping from a Faucet	Rayleigh,	?
	Langhaar, 1951	27 (prob. 19)
	Focken, 1953	15, 50
Weight of Sand Grains Carried by Wind Erosion	Langhaar, 1951	46 (prob. 24)
Maximum Height of a Geyser of Water	Langhaar, 1951	28 (prob 28)
Natural Mode of Oscillation of Frictionless Liquid in an Open Container		
	Rayleigh, 1915	Nature
	Langhaar, 1951	27 (prob. 20)
Frequency of Eddy Shedding from an Open Gate	Langhaar, 1951	28 (prob. 21)
Frequency of Generalized Eddies in a Fluid	Focken, 1953	212-3(prob. 3)
Point of Transition to Turbulent Flow over a Plate	Langhaar, 1951	28 (prob. 25)
Turbulent Flow in a Smooth Circular Pipe	Isaacson, 1975	121-3
Thickness of Turbulent Boundary Layer	Isaacson, 1975	123-4
Frequency of Any Vibration Mode of Liquid Drops	Langhaar, 1951	45 (prob. 17)
Frequency of Vibration of Drops Due to Surface Tension and Viscosity		
	Bridgman, 1922	3-4
	Huntley, 1952	60-1
	Focken, 1953	151 (prob. 4)
	Taylor, 1974	83
Frequency of Vibration of a Nicholson Hydrometer	Huntley, 1952	51-2, 84-6
	Isaacson, 1975	58-9
Frequency of Vibration of Mercury in a U-tube	Huntley, 1952	52-3
Maximum Diameter of a Raindrop	Langhaar, 1951	45 (prob. 20)
Maximum Spin Rate of Liquid Drops	Taylor, 1974	83
Number of Raindrops Hitting a Windshield	Langhaar, 1951	45 (prob. 23)
Height of a Splash Due to Rain Hitting Water	Langhaar, 1951	46 (prob. 26)
Pressure Caused by Underwater Explosions	Langhaar, 1951	70-71
	Isaacson, 1975	160-1, 174-5
Size of Underwater Bubbles	Focken, 1953	152 (prob. 9)

Excess Gaseous Pressure in a Soap Bubble	Huntley, 1952	59-60
Natural Frequency of Thin, Hollow Elastic Spheres (Bubbles) With Internal Gas Pressure		
	Taylor, 1974	153
Buoyant Force on a Flabby Balloon Submerged in Liquid	Taylor, 1974	157
Velocity Distribution of Turbulent Flow Near a Wall	Langhaar, 1951	99-102
Shear In a Turbulent Flow Field	Langhaar, 1951	102-105
Shear of Wind Passing Over Land	Langhaar, 1951	117 (prob. 1)
Shear of Wind Passing Over a Frozen Lake	Langhaar, 1951	117 (prob. 2)
Scaling Effects of Boundary Layers	Langhaar, 1951	105-109
Uniform flow in a Flume	Langhaar, 1951	109
Flow Over A Broad-Crested Weir	Isaacson, 1975	73
Flow of Liquids Through Notches	Duncan, 1953	71-2
Behavior of a Plume of Gas from a Chimney Stack	Isaacson, 1975	163-6

B.3.3 Thermal Systems

1-D Conductive Heat Flow	Duncan, 1953	99-100
3-D Conductive Heat Flow with Capacitance	Duncan, 1953	100-1
	Kline, 1965	71-9, 89-90, 94-8
Periodic Temperature Applied to Semi-Infinite Solid	Duncan, 1953	103
Specific Heat of an Ideal Gas	Focken, 1953	152 (prob. 14)
Carnot Efficiency	Taylor, 1974	30-1
Critical Temperature of Helium-3	Isaacson, 1975	101-103
Specific Heat of Solids and Emission Spectra (Einstein's Example)		
	Focken, 1953	161
Pressure of an Ideal Gas and Rate of Leak into a Vacuum	Focken, 1953	152 (prob. 16)
Clapeyron-Clausius Latent Heat Equation	Focken, 1953	152 (prob. 17)
Heat Transfer to a Flowing Fluid in a Pipe	Langhaar, 1951	122-123
	Focken, 1953	210

Heat Transfer from a Moving Fluid to a Moving Fluid Through a Plane Metal Wall (Weak Variables)		
	Taylor, 1974	57-8
Film Coefficient of Heat Transfer to a Flowing Fluid in a Pipe		
	Huntley, 1952	123
Similarity of Chemical Pilot Scale Plants to Full Scale	Focken, 1953	210
Condensation in a Vertical Pipe	Langhaar, 1951	123-125
Maximum Rate of Evaporation of a Liquid	Focken, 1953	153 (prob. 19)
Transient Heat Transmission of a Body in a Fluid due to Convection (Boussinesq's Problem)		
	Rayleigh, 1915	Nature
	Bridgman, 1922	9-11
	Langhaar, 1951	125-126
	Focken, 1953	130
	Taylor, 1974	82
	Isaacson, 1975	93-95
Transient Heat Flow due to Conduction	Taylor, 1974	82-3
Time Scaling for Transient Heating/Cooling Systems	Taylor, 1974	86
Heat Transfer of an Immersed Body by Forced Convection		
	Huntley, 1952	124-6
Correlation between Drag and Heat Transfer on a Body Falling Through a Fluid Under Gravitational Forces		
	Kline, 1965	54-6, 58-61
Heat Transfer Between an Infinitely Conductive Solid of Given Shape and a Fluid Stream		
	Taylor, 1974	15
Natural Convection	Langhaar, 1951	126-127
	Huntley, 1952	126-8
Natural Convection From a Vertical Plate	Isaacson, 1975	95-96
Similarity of Boiling Liquids	Focken, 1953	205
Rate of Cooling of a Quiescent Gas	Focken, 1953	206
Thermal Conductivity of a Gas	Isaacson, 1975	98-9

Heat Transfer of Different Water-Tube Boilers	Langhaar, 1951	128 (prob. 7)
	Isaacson, 1975	104-5
Heat Transfer of Parallel-Flow Heat Exchangers	Kline, 1965	27-9, 63-5
Thermal Conductivity as a Function of Mass Properties	Bridgman, 1922	92
Jean's Law of the distribution of the spectra of a perfect radiator as a function of temperature		
	Focken, 1953	139
Stress Due to Steady Flow of Heat in the Radial Direction of an Annular Disk		
	Taylor, 1974	45
Stress Due to Heating Deformation on a Pipe	Taylor, 1974	155
Scale Modeling of A/C Systems for the British House of Commons		
	Focken, 1953	190-1
Time to Cook a Homogenous Solid Body of Arbitrary Shape		
	Kline, 1965	25-7, 51-3,

B.3.4 Mechanical Engineering

Loading of General Mechanical Structures	Focken, 1953	187-8
Efficiency of Power Transmission of Meshed Gears	Langhaar, 1951	27 (prob. 18)
Stresses Due to Inertial Loads of Crank Mechanisms	Taylor, 1974	83-4
Friction on a Journal Bearing	Langhaar, 1951	42-43
(Full Analysis)	Taylor, 1974	100-111,154
Maximum Pitch of a Flying Boat During Landing	Langhaar, 1951	45 (prob. 14)
Intensity of Sound Produced by Propellers	Langhaar, 1951	46 (prob. 25)
Small Deflection of Elastic Structures	Langhaar, 1951	91
Large Deflection of Elastic Structures	Langhaar, 1951	79
Deflection of an Archery Bow	Langhaar, 1951	80-81
Deflection of Ductile Beams	Langhaar, 1951	83
Deflection of the Center of Fixed End-Point Beams	Isaacson, 1975	36
Wind Deflection and Tilt of Sails and Buoyies	Taylor, 1974	73-5
Loading Beyond the Yield Point of Materials	Langhaar, 1951	81-82
Wind Loads on Large Windows	Langhaar, 1951	82-83

Vibration Frequency of Beams (Tuning Forks)	Focken, 1953	152 (prob. 8)
	Duncan, 1953	46-8
Deflection of a Membrane in Tension	Duncan, 1953	145
Time Scaling of Mechanical Systems and Mechanism	Taylor, 1974	86
Failure of Riveted Joints	Langhaar, 1951	88
Impact Tests of Vehicles and Structures	Langhaar, 1951	94-95
Mass-Spring	Duncan, 1953	45-6
	Kline, 1965	3,164-9
(Poincare's expansion solution)	Kline, 1965	153-8
(Lighthill's expansion solution)	Kline, 1965	158-162
(WKBJ expansion solution)	Kline, 1965	162-163
Mass-Spring-Damper	Focken, 1953	114
	Taylor, 1974	94
(complete solution)	Kline, 1965	113-117
Vibrating Systems of Mass-Spring-Viscous Damper (Viscosity must Scale with Size)	Taylor, 1974	84-5
2 DOF Undamped Mass-Spring System	Isaacson, 1975	174-4
An Eddy-Current Brake for Absorbing and Measuring Shaft Power (Full Analysis)	Taylor, 1974	97-100
Scaling Laws for Reciprocating IC Engines (Full analysis) (For max power, speed of pistons is constant; Power is proportional to piston area)	Taylor, 1974	111-129
Design Considerations for a Sailboat (Full analysis)	Taylor, 1974	129-144
Performance Laws for Jet Aircraft (Full Analysis)	Taylor, 1974	144-152

B.3.5 Electrical Engineering

Ohm's Law	Isaacson, 1975	139-40
The Electromagnetic Wave Equation	Isaacson, 1975	131-4
Electrical Conductivity of Metals as a Function of Free Electrons, Ratio of Thermal to Electrical Conductivity (Lorenz Law and Wiedman-Franz Laws)	Focken, 1953	154, 172

	Isaacson, 1975	140-1
Ferromagnetism	Langhaar, 1951	137
Thermostats and Governing Pi Parameters	Langhaar, 1951	137-139
Piezoelectric Materials and Pi Parameters	Langhaar, 1951	139-142
Electromagnetic Charge Distribution on a Sphere	Bridgman, 1922	12, 53
Charge Distribution due to Shape Effects of a Conductor	Focken, 1953	142
Skin Effect of High-Frequency Currents	Isaacson, 1975	136-7
	Duncan, 1953	113
Similarity Conditions for Two Different Circuits	Duncan, 1953	110-11
Oscillation of Charge in an L-C Circuit	Bridgman, 1922	77-8
	Focken, 1953	114-5
	Duncan, 1953	109
Rate of Decay of an R-L Circuit	Huntley, 1952	141
	Focken, 1953	143
	Duncan, 1953	109
Rate of Decay of an R-C Circuit	Duncan, 1953	109
Oscillation of RLC Circuit	Isaacson, 1975	137-8
	Duncan, 1953	110
Natural Frequencies of Any Passive Circuit	Taylor, 1974	85-6
Arbitrary RLC Circuits in Current/Voltage/Time Units	Taylor, 1974	28-30
How Frequency of an A/C Electrical System Should Vary with Scale Size		
	Taylor, 1974	87
Space-Charge Limited Current Emitted by Thermionic Surfaces in a Vacuum		
	Focken, 1953	110
Torque on a Magnetic Dipole in a Uniform Magnetic Field	Focken, 1953	142
Power Radiated by Isolated Hertzian Oscillator (Oscillating Electric Dipole)		
	Duncan, 1953	115
Power Radiated by Oscillating Magnetic Dipole (Loop Antenna)		
	Duncan, 1953	115
Space Density of Electromagnetic Energy	Focken, 1953	144

	Isaacson, 1975	138
Scaling Limitations to the Speed of Ultrahigh Frequency Electronics Due to Thermal Effects		
	Focken, 1953	146
	Massey, 1971	82-5
Attraction between Two Plates of a Capacitor	Focken, 1953	152 (prob. 25)
	Isaacson, 1975	135-6,145
Radius of Curvature of an Electron in a Mag. Field	Huntley, 1952	141-3
	Isaacson, 1975	135
Magneto-Fluid Dynamics of a Body Immersed in a Homogenous Viscous Fluid of Constant Density Subject to Magnetic and Electric Fields	Massey, 1971	71

B.3.6 Geological Engineering

Pressure Beneath the Earth's Surface	Isaacson, 1975	79-81
Cooling of a Tunnel Wall	Isaacson, 1975	158-160
Structural Deformation and Folding of Crustal Layers	Focken, 1953	192-194
Energy Released by an Earthquake	Isaacson, 1975	p 38
Erosion of a Riverbed	Isaacson, 1975	161-3
Experimental Prediction of Formation of Mountains and Mountain Ranges		
	Focken, 1953	195
Colossal Impact Behavior of Earth with another Celestial Body		
	Focken, 1953	196
Annual, Periodic Changes in Subsurface Soil Temperature	Isaacson, 1975	96-8

B.3.7 Biological Engineering

Compressive Stress in Legs of Land Animals Varies Linearly with Length		
	Isaacson, 1975	185
Muscular Force Varies as Length Squared	Isaacson, 1975	184
Work Varies as Length Cubed	Isaacson, 1975	184
Jumping Height of Animals is Independent of Size	Isaacson, 1975	184
Efficiency of Motion Varies as Area of Lung/Weight = 1/Length		

	Isaacson, 1975	184
Frequency of Animal Sounds Vary as Length Squared	Isaacson, 1975	185
Speed of Walking Varies as L^(1/2) (Swing time governed by pendulum)		
	Isaacson, 1975	185
Relative Weight of Intake of Food Varies as 1/L	Isaacson, 1975	185
Maximum Speed of Land Animals is Constant (wrong)	Huntley, 1952	41
Characteristic Time (Time to Consume Mass of O2 Equal to Weight ~ Maximum Time Without Feeding Before Starvation) Versus Characteristic Length (length of side of cube of water equal in weight)	Taylor, 1974	88

B.4 Astronomy

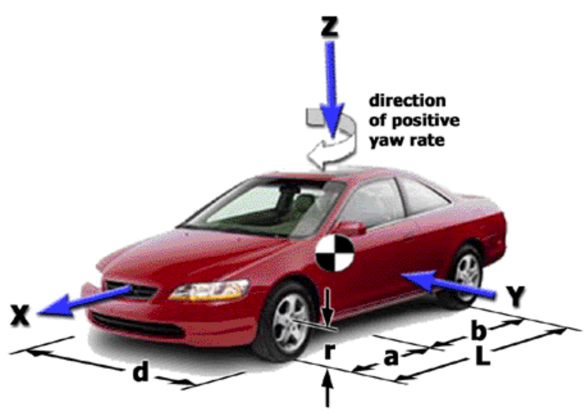
Astronautical Units of Measure	Langhaar, 1951	9
Frequency of Vibration of Spheres Held Together by Gravity (Natural Mode Vibration of a Star)		
	Rayleigh, 1915	Nature
	Langhaar, 1951	25
	Focken, 1953	152 (prob. 6)
	Taylor, 1974	153
Period of Revolution of Two Bodies (Kepler's Third Law)	Bridgman, 1922	5-8
	Huntley, 1952	21-3
	Focken, 1953	38-9, 152
	Duncan, 1953	32-3
	Taylor, 1974	81-2

B.5 Economics

Cost of Wind Tunnels	Langhaar, 1951	12

Appendix C: Vehicle Parameters and References

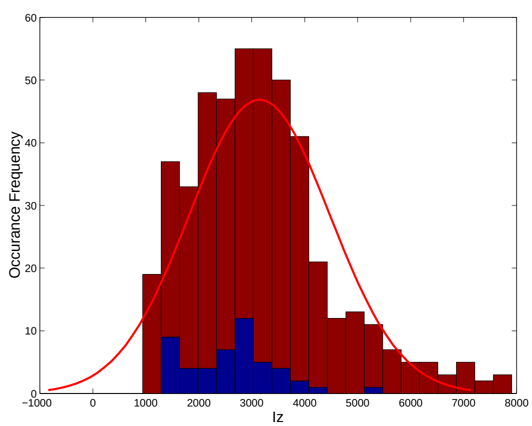
This appendix lists references and parameters for the approximately 700 vehicles used to generate distributions and a measure of an average vehicle behavior. The 8 parameters of interest are given by the bicycle model and they are: (1) *a*, the distance from the C.G. to the front axle, (2) *b,* the distance from the C.G. to the rear axle, (3) *Caf*, the front cornering stiffness, or force produced per unit angle of the front tire, (4) *Car*, the rear cornering stiffness, (5) *Iz*, the moment of inertia of the vehicle about the z-axis, (6) *U*, the velocity of the vehicle in the x-direction, assumed constant (7) *m,* the mass of the vehicle, and (8) *L*, the length of the wheelbase of the vehicle. Several of the geometric parameters are shown below:

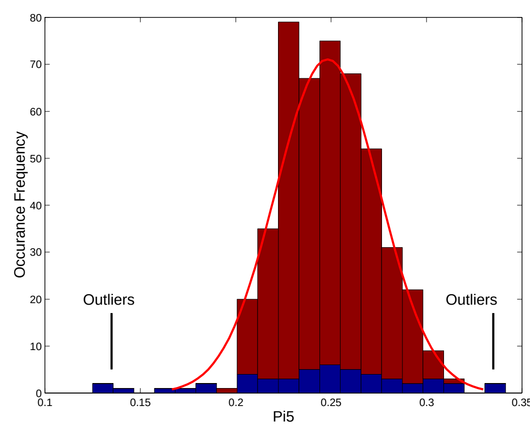


C.1 Plots of Parameter Distributions

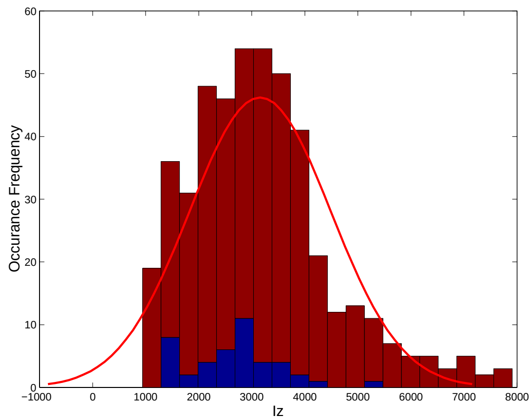
As of August 26, 2002, a total of 73 sets of vehicle parameters have been compiled from a survey of literature, primarily focusing on publications from the Amercan Control Conferences from 1997-2002, the Journal of Vehicle System Dynamics from 1992-2002, and the Symposiums for Advanced Vehicle Control (AVEC) on the years 2000, 1998, 1996, and 1994. Of the 73 sets of vehicles, 21 (29%) of the vehicles are obvious repeats of prior publications. Of the 52 that remain after eliminating repeats, 7 of these (13%) are clearly outliers based on an analysis discussed below. In 2002, a dataset used by the National Highway Transportation and Safety Administration (NHTSA) was dicsovered which includes measured values for approximately 700 vehicles. Unfortunately, the database does not include tire force measurements which are necessary to determine cornering stiffness values.

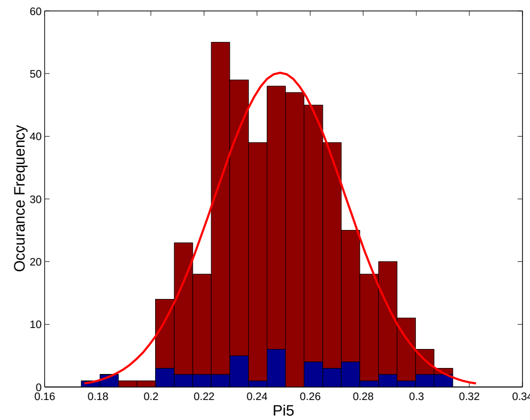
The presence of outliers in the publication set was only determined after including the massive amount of NHTSA data. Additionally, the outliers were certainly not obvious in the parameter distributions, but were only revealed in the distributions of the dimensionless parameters; see Chapter 3 of the thesis for reasons and motivation for conducting an analysis of the dimensionless parameters. Shown below are the plots for the dimensioned (left plot) and dimensionless (right plot) values for the momemnt of inertia. The dimensionless parameter is formed by dividing the moment of inertia by the vehicle mass times the wheel base length squared.



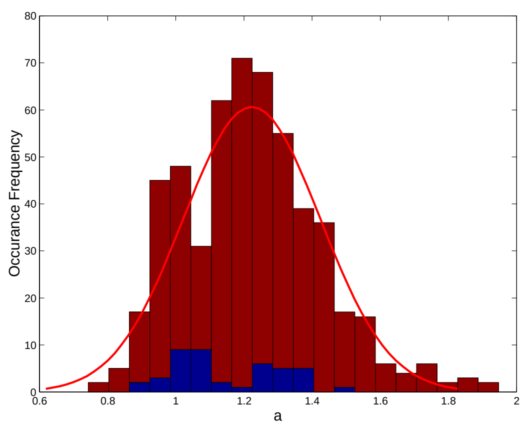


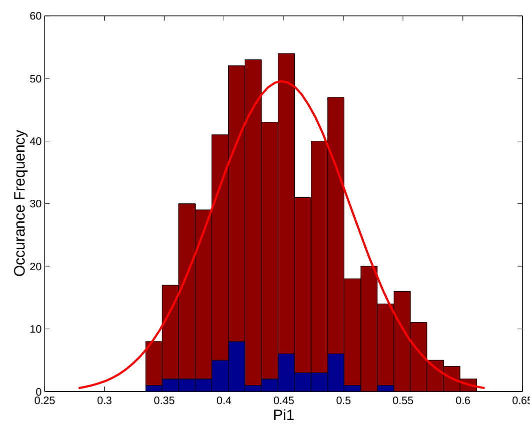
Before the outliers were removed, the standard deviations of the pi5 parameter for the NHTSA data was 0.0234, while the standard deviations for the journal publications is 0.0491. With both datasets combined, the standard deviation becomes 0.0272, and clearly the small set of journal publications are adding a significant variation to the NHTSA data. With the combined dataset, the 3-standard deviation interval is between [0.1666, 0.3297], and vehicle datasets with Iz values outside this interval are considered outliers. To reiterate the scatter present in vehicle control publications, of the approx. 700 vehicles in the NHTSA dataset, none were outliers, yet of the 53 vehicles in control-related publications, 7 were outliers. The Iz and Pi5 plots with outliers removed are shown below:



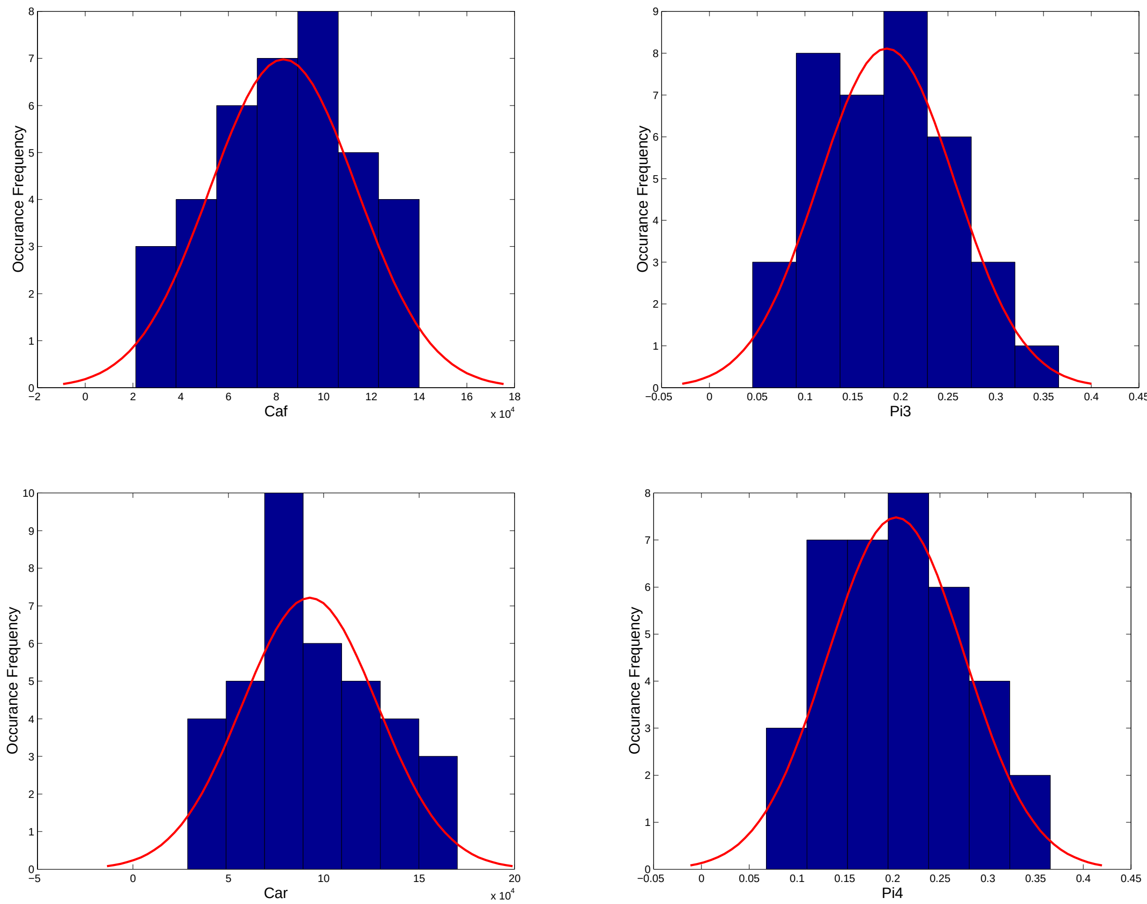


The remaining parameters are now shown alongside their respective dimensionless parameter.

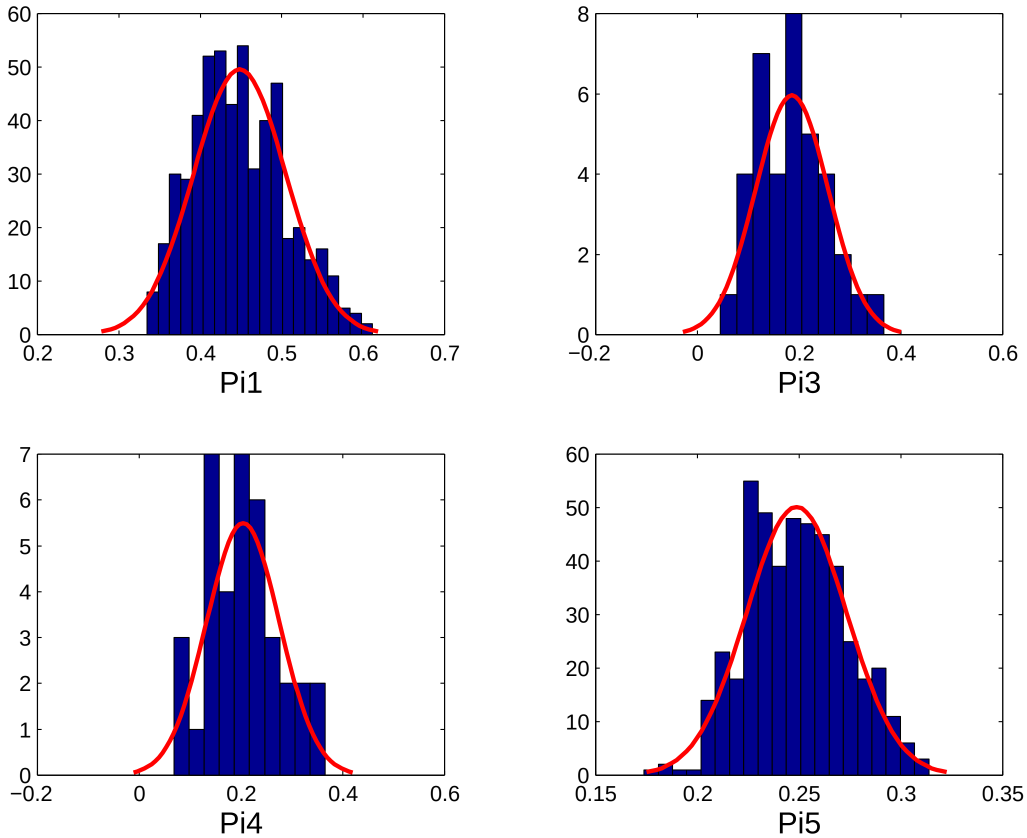




Note that the ‘b’ parameter can be determined from the a parameter and the length of the vehicle. Thus, the distributions of this parameter are omitted. Note that the following parameters, which represent tire forces, do not have measurements in the NHTSA database. Therefore, the sample sizes are much smaller.



Finally, a distribution of all the relevant parameters is given:



We now present the data and references used to generate the above plots.

C.2 Listing of Publications with Bicycle Model Parameters

The listing of sources for the vehicle parameters plotted above are given below, excluding the NHTSA dataset.

Reference	Author	Model Type	Affiliation	Year	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]	Repeat?	Outlier?	
					m	Iz	a	b	Caf	Car	U			
1	Ahring et al	Simulation	Ford AG, Germany	1995	1250.0	2000.0	1.104	1.296	92150	90750	18.0	0	0	
1	Ahring et al	Simulation	Ford AG, Germany	1995	1250.0	2000.0	1.104	1.296	73720	72600	18.0	1	0	
1	Ahring et al	Simulation	Ford AG, Germany	1995	1460.0	2350.0	1.296	1.104	92150	88630	18.0	0	0	
1	Ahring et al	Simulation	Ford AG, Germany	1995	1460.0	2350.0	1.296	1.264	73720	70900	18.0	1	0	
2	Alleyne	Simulation	UIUC	1997	1670.0	2100.0	0.990	1.700	123200	104200	28.0	1	0	
51	Alleyne	Simulation	UIUC	1997	1670.0	2100.0	0.990	1.700	123200	104200	28.0	1	0	
59	Alleyne	Simulation	UIUC	1997	1600.0	3000.0	1.000	1.500	90000	90000	28.0	0	0	
29	Bevly	Full-Sized	Stanford	2001	1640.0	3500.0	1.300	1.500	100000	160000	10.0	0	0	1
3	Bundorf	Simulation	General Motors	1968	1814.0	4203.0	1.520	1.520	21153	31857	26.8	0	0	
40	Chan and Tan	Full-Sized	PATH	1999	1750.0	3217.0	1.060	1.760	34962	65069	20.0	1	0	2
39	Chen and Tan	Full-Sized	PATH	1999	1740.0	3214.0	1.058	1.756	60000	122000	20.0	1	0	
4	Cho and Kim	Simulation	Univ. of Cincinatti	1996	960.0	1600.0	1.000	1.400	28650	28650	50.0	0	0	
52	Cho and Kim	Simulation	Univ. of Cincinatti	1996	960.0	1600.0	1.000	1.400	28650	28650	50.0	0	0	
5	Doniselli	Indy Vehicle	Politecnico di Milano	1996	650.0	850.0	1.630	1.180	140000	180000	32.0	0	0	
43	El-Enswamy	Simulation	Duke	1998	2115.2	1931.0	1.170	1.220	83622	83622	29.1	0	1	
41	Feng	Full-Sized	PATH	1998	1740.0	3214.0	1.058	1.756	58000	120000	20.0	1	0	3
38	Feng	Full-Sized	PATH	1999	1740.0	3214.0	1.058	1.756	70000	180000	20.0	1	0	3
53	Feng	Full-Sized	PATH	2000	1740.0	3214.0	1.058	1.756	35000	90000	20.0	1	0	3
54	Gerdes	Full-Sized	Stanford	2002	1900.0	3100.0	1.387	1.443	140000	170000	NaN	0	0	4
28	Guvenc	Simulation	Istanbul Tech.	2001	1296.0	1750.0	1.250	1.320	84243	95707	50.0	0	0	
6	Harada	Simulation	Nat. Def. Acad.	1996	1690.0	2940.0	1.310	1.370	82200	120600	NaN	0	0	
6	Harada	Simulation	Nat. Def. Acad.	1996	1790.0	3810.0	1.270	1.545	91400	155200	NaN	0	0	
7	Hatipoglu	Full-Sized	OSU Test Vehicle	1998	1569.0	2724.0	1.350	1.370	59600	86600	40.0	0	0	
42	Hingwe and Tomizuka	Full-Sized	PATH	1997	1500.0	2872.0	1.100	1.580	84000	84000	6.0	1	0	
8	Horiuchi	Full-Sized	Honda R & D	1996	1484.8	1333.6	1.163	1.402	78072	84078	20.7	0	1	
32	Huang	Full-Sized	PATH	2001	1740.0	3217.0	1.040	1.760	70000	130000	35.0	1	0	5
55	Jang	Full-Sized	U.C. Davis	2000	1460.0	2743.0	0.930	1.760	NaN	NaN	22.4	0	0	6
55	Jang	Full-Sized	U.C. Davis	2000	1554.7	2778.0	0.950	1.740	NaN	NaN	22.4	0	0	7
55	Jang	Full-Sized	U.C. Davis	2000	1763.5	2995.0	1.070	1.620	NaN	NaN	22.4	0	0	8
55	Jang	Full-Sized	U.C. Davis	2000	1857.9	3282.0	1.170	1.520	NaN	NaN	22.4	0	0	9
60	Langson	Simulation	UIUC	1997	1450.0	2500.0	1.270	1.370	100000	100000	28.0	0	0	
9	LeBlanc	Full-Sized	UMTRI Ford	1995	1814.0	3962.0	1.073	1.620	107462	132880	NaN	0	0	10
56	LeBlanc	Full-Sized	UMTRI Ford	1995	1814.0	3962.0	1.073	1.620	127520	132860	NaN	1	0	10
44	Lee, A. Y.	Simulation	G.M.	1990	1175.0	2618.0	0.946	1.719	96000	84000	NaN	0	0	
57	Lee, Yonggon	Simulation	Purdue	2002	1280.0	2500.0	1.200	1.220	NaN	NaN	NaN	1	0	
10	Lin	Simulation	PATH	1992	1300.0	1630.0	1.000	1.450	65000	54000	NaN	0	0	
58	Lu	Simulation	PATH	2002	1485.0	2872.0	1.100	1.580	84000	84000	30.0	0	0	
35	Mammar	Full-Sized	CEMIF	2000	991.0	1574.0	1.007	1.460	49400	54400	30.0	1	0	
31	Mammar	Simulation	CEMIF	2001	991.0	1574.0	1.007	1.463	115132	115024	51.0	0	0	
11	Matsumoto	Scale	PATH	1992	50.0	5.0	0.472	0.392	2000	2000	2.0	0	0	
46	Modjtahedzadeh and Hess	Full-Sized	U.C. Davis	1993	1814.0	5242.0	1.400	1.710	73091	59840	13.9	0	0	11
46	Modjtahedzadeh and Hess	Full-Sized	U.C. Davis	1993	1400.0	2232.0	1.250	1.420	54563	48030	13.9	0	0	12
12	Nagai	Simulation	Tokyo	1995	1300.0	3000.0	1.000	1.600	44400	43600	22.2	0	1	
13	Nagai	Simulation	Tokyo	1997	1562.0	2630.0	1.104	1.421	42000	64000	27.7	0	0	
14	Nagai	Simulation	Tokyo	1998	1926.0	3685.0	1.264	1.516	48000	82000	27.7	0	0	
15	Palkovic	Simulation	Univ. of Budapest	1992	1200.0	1800.0	1.100	1.200	68000	58000	30.0	1	0	

Reference					Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]		Repeat?	Outlier?	
15	Palkovic	Simulation	Univ. of Budapest	1992	1200.0	1800.0	1.100	1.200	68000	48000	30.0	1	0	
16	Peng	Simulation	PATH	1993	1573.0	2783.0	1.034	1.491	92000	75600	32.0	0	0	
17	Peng	Full-Sized	PATH	1994	1720.0	3250.0	1.140	1.530	80000	80000	32.0	0	0	13
18	Peterson	Full-Sized	Unknown	1996	1457.2	2053.0	1.290	1.280	116000	130000	40.0	0	0	14
18	Peterson	Full-Sized	Unknown	1996	1550.0	2725.0	1.380	1.380	116000	130000	40.0	0	0	15
19	Pilluti	Full-Sized	Ford	1996	1670.0	2100.0	0.990	1.700	123190	104910	27.8	0	0	
61	Pilluti	Full-Sized	Ford	1996	1670.0	2100.0	0.990	1.700	123190	104910	27.8	1	0	
63	Russo	Full-Sized	Fiat, Naples U.	2000	986.0	1320.0	0.883	1.450	NaN	NaN	27.8	0	0	
34	Samadi	Simulation	Irankhodro Co., Iran	2001	1205.0	1600.0	1.080	1.590	NaN	NaN	NaN	0	0	
62	Sharp	Simulation	unknown	2001	1200.0	1500.0	0.920	1.380	120000	80000	NaN	0	0	
20	Shibahata	Full-Sized	Honda R & D	1992	1350.0	NaN	1.000	1.500	NaN	NaN	32.0	0	1	
21	Shiller	Simulation	UCLA	1998	1550.0	3100.0	2.000	2.000	80000	80000	50.0	0	1	
48	Shiotsuka	Simulation	Tokyo Inst. of Tech.	1993	1200.0	1800.0	1.100	1.300	80000	80000	13.8	0	0	
30	Shrivastava	Full-Sized Truck	University of Minnesota	2001	9053.0	52161.0	2.590	4.700	130000	130000	NaN	0	0	
45	Sivashankar and Ulsoy	Full-Sized	Ford, U. Mich.	1998	2000.0	2712.0	1.040	1.650	137510	117800	20.0	0	0	
22	Smith and Benton	Simulation	LSU	1996	1280.0	2500.0	1.203	1.217	60000	60000	NaN	0	1	
23	Smith and Starkey	Simulation	LSU and Purdue	1995	1280.0	2500.0	1.203	1.217	40000	40000	30.0	1	0	
23	Smith and Starkey	Simulation	LSU and Purdue	1995	1298.9	1627.0	1.000	1.454	58000	58000	30.0	0	0	
49	Smith and Starkey	Simulation	LSU and Purdue	1994	1280.0	2500.0	1.203	1.217	60000	60000	30.0	1	0	
24	Sridar	Simulation	IIT Kanpur, India	1995	1500.0	2500.0	1.250	1.500	50000	64000	NaN	0	0	
25	Tagawa et al	Simulation	Tokyo and Honda	1996	1640.0	2720.0	1.105	1.345	66040	111660	NaN	0	0	
37	Tan and Chan	Full-Sized	PATH	2000	1750.0	3217.0	1.060	1.760	69924	130138	20.0	1	0	
37	Tan and Chan	Full-Sized	PATH	2000	1510.0	3452.0	1.300	1.410	77574	71964	20.0	0	0	
65	Tan and Chan	Full-Sized	PATH	2002	1740.0	3214.0	1.058	1.756	58000	120000	20.0	0	0	16
26	Unnyelioglu	Simulation	Ohio State CITR	1997	1900.0	1750.0	1.000	1.700	117000	105000	50.0	0	1	
66	Venhovens	Full-Sized	BMW Munich	1999	2000.0	3500.0	1.400	1.400	70000	140000	NaN	0	0	
27	Will and Zak	Simulation	Purdue	1997	1280.0	2500.0	1.203	1.217	40000	40000	18.3	1	0	
33	Zhang et al	Simulation	Universite Picardie	2001	1480.0	2350.0	1.050	1.630	135000	95000	10.0	0	0	

C.3 Listing of All Unique, Non-Outlier Vehicles

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
1	Ford AG, Germany	Simulation	1995	1250.00	2000.00	1.104	1.296	92150	90750
1	Ford AG, Germany	Simulation	1995	1460.00	2350.00	1.296	1.104	92150	88630
59	UIUC	Simulation	1997	1600.00	3000.00	1.000	1.500	90000	90000
29	Stanford, Mercedes E-class sedan	Full-Sized	2001	1640.00	3500.00	1.300	1.500	100000	160000
3	General Motors	Simulation	1968	1814.00	4203.00	1.520	1.520	21153	31857
4	Univ. of Cincinatti	Simulation	1996	960.00	1600.00	1.000	1.400	28650	28650
52	Univ. of Cincinatti	Simulation	1996	960.00	1600.00	1.000	1.400	28650	28650
54	Stanford, Mercedes E320 Sedan + 4 passengers + 40 psi all tires	Full-Sized	2002	1900.00	3100.00	1.387	1.443	140000	170000
28	Istanbul Technical University	Simulation	2001	1296.00	1750.00	1.250	1.320	84243	95707
6	National Defense Academy	Simulation	1996	1690.00	2940.00	1.310	1.370	82200	120600
6	National Defense Academy	Simulation	1996	1790.00	3810.00	1.270	1.545	91400	155200
7	OSU Test Vehicle	Full-Sized	1998	1569.00	2724.00	1.350	1.370	59600	86600
55	U.C. Davis, Ford Tarus - Empty	Full-Sized	2000	1460.00	2743.00	0.930	1.760	NaN	NaN
55	U.C. Davis, Ford Tarus - Driver only	Full-Sized	2000	1554.70	2778.00	0.950	1.740	NaN	NaN
55	U.C. Davis, Ford Tarus - Driver + 3 Passengers	Full-Sized	2000	1763.50	2995.00	1.070	1.620	NaN	NaN
55	U.C. Davis, Ford Taurus - Driver + 3 Passengers + Rear Cargo	Full-Sized	2000	1857.90	3282.00	1.170	1.520	NaN	NaN
60	UIUC	Simulation	1997	1450.00	2500.00	1.270	1.370	100000	100000
9	UMTRI 1994 Ford Taurus SHO	Full-Sized	1995	1814.00	3962.00	1.073	1.620	107462	132880
44	General Motors Research Laboratories	Simulation	1990	1175.00	2618.00	0.946	1.719	96000	84000
10	PATH	Simulation	1992	1300.00	1630.00	1.000	1.450	65000	54000
58	PATH	Simulation	2002	1485.00	2872.00	1.100	1.580	84000	84000
31	CEMIF, Universite d'Evry val d'Essone, France	Simulation	2001	991.00	1574.00	1.007	1.463	115132	115024
46	U.C. DavisFull-sized sedan	Full-Sized	1993	1814.00	5242.00	1.400	1.710	73091	59840
46	U.C. Davis,Compact car	Full-Sized	1993	1400.00	2232.00	1.250	1.420	54563	48030
13	Tokyo	Simulation	1997	1562.00	2630.00	1.104	1.421	42000	64000
14	Tokyo	Simulation	1998	1926.00	3685.00	1.264	1.516	48000	82000
16	PATH	Simulation	1993	1573.00	2783.00	1.034	1.491	92000	75600
17	PATH Pontiac 6000	Full-Sized	1994	1720.00	3250.00	1.140	1.530	80000	80000
18	BMW 325i Lead car	Full-Sized	1996	1457.20	2053.00	1.290	1.280	116000	130000
18	BMW 518i	Full-Sized	1996	1550.00	2725.00	1.380	1.380	116000	130000
19	Ford	Full-Sized	1996	1670.00	2100.00	0.990	1.700	123190	104910
63	Fiat and Naples University	Full-Sized	2000	986.00	1320.00	0.883	1.450	NaN	NaN
34	Irankhodro Co., Iran	Simulation	2001	1205.00	1600.00	1.080	1.590	NaN	NaN
62	unknown	Simulation	2001	1200.00	1500.00	0.920	1.380	120000	80000
48	Tokyo Inst. of Tech.	Simulation	1993	1200.00	1800.00	1.100	1.300	80000	80000
45	Ford Motor Company, University of Michagan respectively	Full-Sized	1998	2000.00	2712.00	1.040	1.650	137510	117800
23	LSU and Purdue	Simulation	1995	1298.90	1627.00	1.000	1.454	58000	58000
24	IIT Kanpur, India	Simulation	1995	1500.00	2500.00	1.250	1.500	50000	64000
25	Tokyo and Honda	Simulation	1996	1640.00	2720.00	1.105	1.345	66040	111660
37	PATH	Full-Sized	2000	1510.00	3452.00	1.300	1.410	77574	71964
65	PATH 1996/1997 Buick LeSabre	Full-Sized	2002	1740.00	3214.00	1.058	1.756	58000	120000
66	BMW Munich	Full-Sized	1999	2000.00	3500.00	1.400	1.400	70000	140000
33	Universite Picardie	Simulation	2001	1480.00	2350.00	1.050	1.630	135000	95000
67	1984 Audi Quattro 4000,4S,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1239.65	2352.00	1.124	1.396	NaN	NaN
"	1980 BMW 320i,2S,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1980	1199.39	NaN	1.175	1.416	NaN	NaN
"	1986 BMW 325i,2S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1251.48	2027.00	1.201	1.369	NaN	NaN
"	1986 Buick Century Estate,SW,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1518.55	3162.00	1.120	1.547	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1506.32	3073.00	1.043	1.771	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1496.33	3041.00	1.040	1.774	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: NaN	Full-Sized	1986	1492.25	2991.00	1.050	1.764	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1491.85	3005.00	1.035	1.779	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1500.92	3045.00	1.037	1.777	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1496.33	2977.00	1.040	1.774	NaN	NaN
"	1986 Buick Electra,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1506.32	3103.00	1.045	1.769	NaN	NaN
"	1980 Buick LeSabre_S/C,2S,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1980	1775.23	NaN	1.322	1.624	NaN	NaN
"	1986 Buick Skylark,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1261.88	2082.00	0.942	1.674	NaN	NaN
"	1991 Chevrolet 1500 Silverado,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1895.82	4924.00	1.398	1.942	NaN	NaN
"	1979 Chevrolet 20 Beauville,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1979	2271.76	NaN	1.420	1.742	NaN	NaN
"	1998 Chevrolet Astro,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2030.68	3973.00	1.298	1.523	NaN	NaN
"	1998 Chevrolet Astro,VN,7 passengers,ballast of 1300,RWD,fuel tank: F	Full-Sized	1998	2612.13	4876.00	1.543	1.278	NaN	NaN
"	1998 Chevrolet Astro,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2077.47	4342.00	1.303	1.518	NaN	NaN
"	1987 Chevrolet Astro Van,VN,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1987	2390.52	NaN	1.469	1.368	NaN	NaN
"	1987 Chevrolet Astro Van,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1776.15	NaN	1.236	1.601	NaN	NaN
"	1987 Chevrolet Astro Van,VN,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	2008.26	NaN	1.231	1.606	NaN	NaN
"	1988 Chevrolet Astro Van,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	1797.86	3413.00	1.217	1.628	NaN	NaN
"	1988 Chevrolet Astro Van,VN,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1988	1752.09	3390.00	1.192	1.653	NaN	NaN
"	1988 Chevrolet Astro Van,VN,6 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	2173.80	3836.00	1.355	1.490	NaN	NaN
"	1998 Chevrolet Blazer,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1963.00	3415.00	1.216	1.502	NaN	NaN
"	1998 Chevrolet Blazer,MP,5 passengers,ballast of 1330,4WD,fuel tank: F	Full-Sized	1998	2400.20	3864.00	1.373	1.345	NaN	NaN
"	1982 Chevrolet C-10 Blazer,MP,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1982	1870.85	3980.00	1.308	1.397	NaN	NaN
"	1982 Chevrolet C-10 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1982	1872.68	4324.00	1.445	1.908	NaN	NaN
"	1988 Chevrolet C-10 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1988	1853.21	3756.00	1.360	1.625	NaN	NaN
"	1987 Chevrolet C-15 pickup,PU,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	2098.98	5364.00	1.440	1.913	NaN	NaN
"	1987 Chevrolet C-15 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1855.05	4858.00	1.403	1.950	NaN	NaN
"	1981 Chevrolet C-20 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	2240.88	5959.00	1.543	1.784	NaN	NaN
"	1981 Chevrolet C-20 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	2212.74	NaN	1.540	1.800	NaN	NaN
"	1998 Chevrolet C1500,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1932.11	4705.00	1.418	1.922	NaN	NaN
"	1998 Chevrolet C1500,PU,3 passengers,ballast of 6709,RWD,fuel tank: F	Full-Sized	1998	2766.16	6327.00	1.797	1.543	NaN	NaN
"	1998 Chevrolet C1500,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1995.62	5331.00	1.415	1.925	NaN	NaN
"	1983 Chevrolet Caprice,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1983	1548.01	3796.00	1.296	1.638	NaN	NaN
"	1984 Chevrolet Caprice Classic,SW,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1977.88	5241.00	1.535	1.437	NaN	NaN
"	1983 Chevrolet Cavalier,SW,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1255.15	2131.00	0.955	1.623	NaN	NaN
"	1986 Chevrolet Cavalier,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1233.33	NaN	0.943	1.648	NaN	NaN
"	1983 Chevrolet Chevette Scooter,3H,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1983	997.55	NaN	1.108	1.305	NaN	NaN
"	1978 Chevrolet K-10 Blazer,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1978	2259.43	4613.00	1.318	1.387	NaN	NaN
"	1982 Chevrolet K-20 pickup,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1982	2227.73	NaN	1.415	1.922	NaN	NaN
"	1985 Chevrolet K-20 pickup,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	2552.80	6465.00	1.412	1.915	NaN	NaN
"	1985 Chevrolet K-5 Blazer,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	2271.76	NaN	1.306	1.386	NaN	NaN
"	1991 Chevrolet K1500 pickup,PU,3 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2148.42	4106.00	1.215	1.789	NaN	NaN
"	1991 Chevrolet K1500 pickup,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2002.85	4037.00	1.201	1.803	NaN	NaN
"	1998 Chevrolet Lumina,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1583.49	2952.00	1.004	1.732	NaN	NaN
"	1998 Chevrolet Lumina,4S,6 passengers,ballast of 449,FWD,fuel tank: F	Full-Sized	1998	2004.18	3553.00	1.227	1.509	NaN	NaN
"	1998 Chevrolet Lumina,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1636.60	3326.00	1.021	1.715	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,7 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	2176.45	4147.00	1.404	1.390	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1648.73	3544.00	1.162	1.632	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1990 Chevrolet Lumina APV,VN,2 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1728.54	3300.00	1.099	1.695	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1724.87	3515.00	1.162	1.632	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,2 passengers,ballast of 4226,FWD,fuel tank: F	Full-Sized	1990	2152.91	4136.00	1.389	1.405	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1652.29	3323.00	1.093	1.701	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1574.31	3379.00	1.094	1.700	NaN	NaN
"	1990 Chevrolet Lumina APV,VN,2 passengers,ballast of 4226,FWD,fuel tank: F	Full-Sized	1990	2150.66	4158.00	1.393	1.401	NaN	NaN
"	1995 Chevrolet Lumina LS,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1995	1593.78	3130.00	0.973	1.758	NaN	NaN
"	1981 Chevrolet Luv,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	1285.93	2721.00	1.257	1.745	NaN	NaN
"	1998 Chevrolet Metro,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	879.41	1102.00	0.965	1.400	NaN	NaN
"	1998 Chevrolet Metro,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	941.59	1426.00	0.979	1.386	NaN	NaN
"	1983 Chevrolet S-10 Blazer,MP,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1983	1392.05	2500.00	1.251	1.292	NaN	NaN
"	1983 Chevrolet S-10 Blazer,MP,4 passengers,ballast of 2224,RWD,fuel tank: 1/2	Full-Sized	1983	1974.31	NaN	1.419	1.146	NaN	NaN
"	1983 Chevrolet S-10 Blazer,MP,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1983	1559.84	NaN	1.275	1.290	NaN	NaN
"	1984 Chevrolet S-10 Blazer,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1589.30	2798.00	1.162	1.378	NaN	NaN
"	1984 Chevrolet S-10 Blazer,MP,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1458.21	2702.00	1.257	1.283	NaN	NaN
"	1989 Chevrolet S-10 Blazer,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1989	1789.30	3187.00	1.209	1.356	NaN	NaN
"	1992 Chevrolet S-10 Blazer,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1728.54	3245.00	1.209	1.349	NaN	NaN
"	1986 Chevrolet S-10 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1201.12	2208.00	1.181	1.562	NaN	NaN
"	1986 Chevrolet S-10 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1519.06	NaN	1.399	1.586	NaN	NaN
"	1986 Chevrolet S-10 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1432.82	NaN	1.205	1.792	NaN	NaN
"	1991 Chevrolet S-10 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1275.94	2776.00	1.266	1.731	NaN	NaN
"	1992 Chevrolet S-10 pickup,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1466.46	2576.00	1.031	1.727	NaN	NaN
"	1986 Chevrolet S-10 Tahoe,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1986	1723.04	NaN	1.288	1.849	NaN	NaN
"	1987 Chevrolet S-10 Tahoe,PU,NaN passengers,ballast of Lt Ld,4WD,fuel tank: F	Full-Sized	1987	1830.48	3594.00	1.293	1.831	NaN	NaN
"	1987 Chevrolet S-10 Tahoe,PU,NaN passengers,ballast of GVWR,4WD,fuel tank: F	Full-Sized	1987	2301.63	4567.00	1.595	1.529	NaN	NaN
"	1987 Chevrolet S-10 Tahoe,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1605.20	3323.00	1.253	1.871	NaN	NaN
"	1998 Chevrolet S10,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1441.79	2477.00	1.169	1.581	NaN	NaN
"	1998 Chevrolet S10,PU,3 passengers,ballast of 3065,RWD,fuel tank: F	Full-Sized	1998	1904.59	3169.00	1.416	1.334	NaN	NaN
"	1998 Chevrolet S10,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1495.01	2897.00	1.175	1.575	NaN	NaN
"	1992 Chevrolet Sportside K-10 pickup,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	2038.23	4045.00	1.179	1.806	NaN	NaN
"	1998 Chevrolet Suburban,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2666.97	7582.00	1.721	1.618	NaN	NaN
"	1998 Chevrolet Tahoe,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2536.90	5445.00	1.441	1.536	NaN	NaN
"	1998 Chevrolet Tahoe,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2601.83	6004.00	1.446	1.531	NaN	NaN
"	1998 Chevrolet Tracker,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1192.86	1416.00	1.005	1.195	NaN	NaN
"	1998 Chevrolet Tracker,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1256.17	1743.00	1.011	1.189	NaN	NaN
"	1998 Chevrolet Venture,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1823.24	4065.00	1.255	1.792	NaN	NaN
"	1998 Chevrolet Venture,VN,7 passengers,ballast of 1255,FWD,fuel tank: F	Full-Sized	1998	2399.90	5212.00	1.586	1.461	NaN	NaN
"	1985 Chrysler LeBaron,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1237.92	2160.00	0.990	1.633	NaN	NaN
"	1987 Chrysler LeBaron,2S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1219.78	2110.00	0.967	1.583	NaN	NaN
"	1979 Datsun 210,SW,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1979	1026.10	1739.00	1.131	1.206	NaN	NaN
"	1979 Datsun 280ZX,3H,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1979	1295.01	2058.00	1.157	1.170	NaN	NaN
"	1981 Datsun 510,SW,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	1174.41	NaN	1.188	1.225	NaN	NaN
"	1974 Datsun B210,3H,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1974	945.46	1527.00	1.102	1.242	NaN	NaN
"	1981 Datsun pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	1159.94	2098.00	1.140	1.425	NaN	NaN
"	1998 Dodge 1500,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2265.44	5907.00	1.478	2.040	NaN	NaN
"	1998 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1730.38	3508.00	1.189	1.696	NaN	NaN
"	1998 Dodge Caravan,VN,7 passengers,ballast of 863,FWD,fuel tank: F	Full-Sized	1998	2267.28	4463.00	1.500	1.385	NaN	NaN
"	1998 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1786.14	3923.00	1.199	1.686	NaN	NaN

Reference		Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1987 Dodge Caravan,VN,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1547.60	3202.00	1.190	1.655	NaN	NaN	
"	1987 Dodge Caravan,VN,NaN passengers,ballast of GVWR,FWD,fuel tank: F	Full-Sized	1987	2203.67	3633.00	1.439	1.406	NaN	NaN	
"	1987 Dodge Caravan,VN,NaN passengers,ballast of Lt Ld,FWD,fuel tank: F	Full-Sized	1987	1745.26	3357.00	1.205	1.640	NaN	NaN	
"	1988 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1988	1552.60	3090.00	1.218	1.627	NaN	NaN	
"	1988 Dodge Caravan,VN,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	1532.62	3133.00	1.180	1.670	NaN	NaN	
"	1990 Dodge Caravan,VN,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1580.73	3736.00	1.260	1.766	NaN	NaN	
"	1991 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1667.79	3427.00	1.074	1.771	NaN	NaN	
"	1991 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1668.20	3468.00	1.175	1.670	NaN	NaN	
"	1991 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1665.04	3439.00	1.180	1.678	NaN	NaN	
"	1992 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1672.68	3433.00	1.154	1.691	NaN	NaN	
"	1992 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1503.57	3051.00	1.097	1.761	NaN	NaN	
"	1992 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1753.01	3552.00	1.148	1.722	NaN	NaN	
"	1992 Dodge Caravan,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1654.13	3365.00	1.201	1.669	NaN	NaN	
"	1989 Dodge Caravan C/V,VN,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1989	1592.46	3268.00	1.232	1.615	NaN	NaN	
"	1989 Dodge Caravan C/V,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1989	1795.62	NaN	1.328	1.695	NaN	NaN	
"	1989 Dodge Colt,3H,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1989	1098.27	1673.00	1.004	1.379	NaN	NaN	
"	1998 Dodge Dakota,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1817.74	4271.00	1.353	1.973	NaN	NaN	
"	1998 Dodge Dakota,PU,5 passengers,ballast of 2344,RWD,fuel tank: F	Full-Sized	1998	2358.00	4960.00	1.582	1.744	NaN	NaN	
"	1987 Dodge Dakota,PU,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1987	1845.46	3212.00	1.419	1.426	NaN	NaN	
"	1987 Dodge Dakota,PU,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	1496.33	2972.00	1.302	1.543	NaN	NaN	
"	1987 Dodge Dakota,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1278.70	2452.00	1.245	1.600	NaN	NaN	
"	1991 Dodge Dakota,PU,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1771.15	4194.00	1.245	1.902	NaN	NaN	
"	1992 Dodge Dakota,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1744.34	4329.00	1.346	1.981	NaN	NaN	
"	1978 Dodge Diplomat,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1978	1649.13	3904.00	1.239	1.621	NaN	NaN	
"	1998 Dodge Durango,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2200.82	4409.00	1.296	1.640	NaN	NaN	
"	1998 Dodge Durango,MP,7 passengers,ballast of 2478,4WD,fuel tank: F	Full-Sized	1998	2902.14	5849.00	1.642	1.294	NaN	NaN	
"	1989 Dodge Dynasty LE,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1989	1559.33	2728.00	0.967	1.692	NaN	NaN	
"	1985 Dodge Lancer,5H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1233.33	2236.00	0.982	1.637	NaN	NaN	
"	1998 Dodge Neon,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1243.53	1945.00	0.954	1.688	NaN	NaN	
"	1998 Dodge Neon,4S,5 passengers,ballast of 437,FWD,fuel tank: F	Full-Sized	1998	1589.40	2340.00	1.178	1.464	NaN	NaN	
"	1998 Dodge Neon,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1301.02	2307.00	0.963	1.679	NaN	NaN	
"	1983 Dodge Omni,5H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1058.31	1649.00	0.975	1.540	NaN	NaN	
"	1983 Dodge Omni,5H,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1983	1022.53	1599.00	0.932	1.583	NaN	NaN	
"	1983 Dodge Omni,5H,2 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1129.56	1690.00	0.986	1.529	NaN	NaN	
"	1983 Dodge Omni,5H,4 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1273.70	1813.00	1.107	1.408	NaN	NaN	
"	1987 Dodge Raider,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1523.55	2318.00	1.168	1.182	NaN	NaN	
"	1989 Dodge Raider,MP,4 passengers,ballast of 2335,4WD,fuel tank: F	Full-Sized	1989	2198.67	3065.00	1.441	0.915	NaN	NaN	
"	1989 Dodge Raider,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1989	1964.32	2643.00	1.293	1.063	NaN	NaN	
"	1989 Dodge Raider,MP,4 passengers,ballast of 2335,4WD,fuel tank: F	Full-Sized	1989	2198.67	3074.00	1.441	0.915	NaN	NaN	
"	1989 Dodge Raider,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1989	1736.70	2527.00	1.211	1.145	NaN	NaN	
"	1981 Dodge Ram,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	1610.60	4211.00	1.341	1.986	NaN	NaN	
"	1987 Dodge Ram B-150,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	2047.71	4589.00	1.231	1.550	NaN	NaN	
"	1987 Dodge Ram B-150,VN,8 passengers,ballast of 1557,RWD,fuel tank: F	Full-Sized	1987	2726.50	5708.00	1.507	1.274	NaN	NaN	
"	1987 Dodge Ram B-150,VN,8 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	2569.22	5136.00	1.392	1.389	NaN	NaN	
"	1987 Dodge Ram B-150,VN,8 passengers,ballast of 1557,RWD,fuel tank: F	Full-Sized	1987	2726.50	5744.00	1.507	1.274	NaN	NaN	
"	1991 Dodge Ram D-150,PU,3 passengers,ballast of 4226,RWD,fuel tank: F	Full-Sized	1991	2494.39	7098.00	1.757	1.570	NaN	NaN	
"	1991 Dodge Ram D-150,PU,3 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	2049.95	5081.00	1.349	1.978	NaN	NaN	
"	1991 Dodge Ram D-150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1903.57	5038.00	1.340	1.987	NaN	NaN	

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1991 Dodge Ram D-150,PU,3 passengers,ballast of 4226,RWD,fuel tank: F	Full-Sized	1991	2494.39	7064.00	1.757	1.570	NaN	NaN
"	1991 Dodge Ramcharger ,MP,5 passengers,ballast of 1112,4WD,fuel tank: F	Full-Sized	1991	2719.27	5671.00	1.482	1.217	NaN	NaN
"	1991 Dodge Ramcharger ,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2311.21	4911.00	1.300	1.399	NaN	NaN
"	1991 Dodge Ramcharger ,MP,5 passengers,ballast of 1112,4WD,fuel tank: F	Full-Sized	1991	2719.27	5710.00	1.482	1.217	NaN	NaN
"	1991 Dodge Ramcharger ,MP,5 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2604.99	5221.00	1.401	1.298	NaN	NaN
"	1988 Ford Aerostar,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	1605.20	3068.00	1.278	1.738	NaN	NaN
"	1989 Ford Aerostar,VN,7 passengers,ballast of 890,RWD,fuel tank: 1/2	Full-Sized	1989	2228.64	NaN	1.566	1.457	NaN	NaN
"	1991 Ford Aerostar,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1785.63	3760.00	1.382	1.636	NaN	NaN
"	1992 Ford Aerostar,VN,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1817.33	3410.00	1.263	1.760	NaN	NaN
"	1986 Ford Aerostar XL,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1710.40	3190.00	1.274	1.736	NaN	NaN
"	1989 Ford Aerostar XL,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1989	1741.18	NaN	1.283	1.740	NaN	NaN
"	1992 Ford Aerostar long,VN,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1900.31	3932.00	1.359	1.666	NaN	NaN
"	1978 Ford Bronco,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1978	2418.65	4853.00	1.318	1.324	NaN	NaN
"	1988 Ford Bronco Custom,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	2190.11	NaN	1.396	1.277	NaN	NaN
"	1984 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1601.02	2539.00	1.191	1.207	NaN	NaN
"	1984 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1657.59	2881.00	1.189	1.209	NaN	NaN
"	1983 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1983	1664.12	NaN	1.093	1.310	NaN	NaN
"	1983 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1983	1664.12	NaN	1.093	1.310	NaN	NaN
"	1985 Ford Bronco II,MP,4 passengers,ballast of 1446,4WD,fuel tank: 1/2	Full-Sized	1985	1944.75	NaN	1.274	1.126	NaN	NaN
"	1985 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	1605.20	NaN	1.139	1.261	NaN	NaN
"	1985 Ford Bronco II,MP,4 passengers,ballast of 111,4WD,fuel tank: 1/2	Full-Sized	1985	1786.14	NaN	1.208	1.192	NaN	NaN
"	1987 Ford Bronco II,MP,0 passengers,ballast of 0,RWD,fuel tank: F?	Full-Sized	1987	1478.19	2357.00	1.194	1.194	NaN	NaN
"	1988 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: E	Full-Sized	1988	1709.48	2628.00	1.246	1.144	NaN	NaN
"	1988 Ford Bronco II,MP,1 passengers,ballast of 0,4WD,fuel tank: E	Full-Sized	1988	1727.62	2603.00	1.242	1.151	NaN	NaN
"	1989 Ford Bronco II XL,MP,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1989	1780.63	2653.00	1.314	1.074	NaN	NaN
"	1983 Ford Bronco XLT,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1983	2162.90	4377.00	1.337	1.343	NaN	NaN
"	1998 Ford Club Wagon,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2528.95	6722.00	1.576	1.929	NaN	NaN
"	1998 Ford Club Wagon,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2589.09	7364.00	1.589	1.916	NaN	NaN
"	1985 Ford E150,VN,8 passengers,ballast of 2113,RWD,fuel tank: 1/2	Full-Sized	1985	2995.01	NaN	1.871	1.634	NaN	NaN
"	1985 Ford E150,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	2236.39	NaN	1.678	1.834	NaN	NaN
"	1985 Ford E150,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	2295.31	6536.00	1.662	1.856	NaN	NaN
"	1985 Ford E150,VN,4 passengers,ballast of 1557,RWD,fuel tank: F	Full-Sized	1985	2687.97	6926.00	1.735	1.783	NaN	NaN
"	1987 Ford E150,VN,0 passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	2267.18	6270.00	1.475	1.979	NaN	NaN
"	1992 Ford E150,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	2256.78	6248.00	1.543	1.972	NaN	NaN
"	1978 Ford E150 ,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1978	2057.70	4590.00	1.404	1.753	NaN	NaN
"	1988 Ford E150 Club Wag XLT,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	2412.74	7028.00	1.701	1.817	NaN	NaN
"	1977 Ford E250,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1977	2186.44	6075.00	1.446	1.737	NaN	NaN
"	1987 Ford E250,VN,0 passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	2285.32	NaN	1.572	1.933	NaN	NaN
"	1985 Ford Escort,2S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1006.63	1545.00	0.830	1.563	NaN	NaN
"	1986 Ford Escort L,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1099.59	NaN	0.894	1.487	NaN	NaN
"	1986 Ford Escort XR3i,2S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1038.33	1519.00	0.965	1.435	NaN	NaN
"	1998 Ford Expedition,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2637.21	5639.00	1.459	1.566	NaN	NaN
"	1998 Ford Expedition,MP,7 passengers,ballast of 1753,4WD,fuel tank: F	Full-Sized	1998	3264.93	6859.00	1.708	1.317	NaN	NaN
"	1998 Ford Explorer,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2017.53	3682.00	1.295	1.532	NaN	NaN
"	1998 Ford Explorer,MP,5 passengers,ballast of 1005,4WD,fuel tank: F	Full-Sized	1998	2421.51	4154.00	1.441	1.386	NaN	NaN
"	1998 Ford Explorer,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	2049.85	3996.00	1.291	1.536	NaN	NaN
"	1992 Ford Explorer,MP,0 passengers,ballast of NaN,RWD,fuel tank: F	Full-Sized	1992	2017.84	4042.00	1.442	1.403	NaN	NaN
"	1992 Ford Explorer Sport,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1905.81	3256.00	1.224	1.367	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1991 Ford Explorer XL,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1970.64	3754.00	1.325	1.507	NaN	NaN
"	1991 Ford Explorer XL,MP,5 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2264.93	3887.00	1.420	1.425	NaN	NaN
"	1991 Ford Explorer XL,MP,5 passengers,ballast of 1268,4WD,fuel tank: F	Full-Sized	1991	2394.60	4292.00	1.516	1.329	NaN	NaN
"	1991 Ford Explorer XL,MP,5 passengers,ballast of 1268,4WD,fuel tank: F	Full-Sized	1991	2394.60	4258.00	1.516	1.329	NaN	NaN
"	1991 Ford Explorer XL,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1966.06	3665.00	1.329	1.516	NaN	NaN
"	1982 Ford F100,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1982	1550.76	3246.00	1.265	1.720	NaN	NaN
"	1998 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	2039.25	5375.00	1.465	2.052	NaN	NaN
"	1984 Ford F150,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1860.96	4527.00	1.333	2.045	NaN	NaN
"	1985 Ford F150,PU,NaN passengers,ballast of Lt Ld,4WD,fuel tank: F	Full-Sized	1985	2214.58	NaN	1.527	1.864	NaN	NaN
"	1985 Ford F150,PU,NaN passengers,ballast of GVWR,4WD,fuel tank: F	Full-Sized	1985	2669.83	NaN	1.867	1.524	NaN	NaN
"	1985 Ford F150,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	2003.26	NaN	1.529	1.862	NaN	NaN
"	1987 Ford F150,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1721.20	NaN	1.145	1.814	NaN	NaN
"	1987 Ford F150,PU,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1987	2235.47	5502.00	1.559	1.408	NaN	NaN
"	1987 Ford F150,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1718.55	3425.00	1.146	1.821	NaN	NaN
"	1987 Ford F150,PU,0 passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	1918.04	3572.00	1.185	1.782	NaN	NaN
"	1987 Ford F150,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1700.41	3428.00	1.139	1.828	NaN	NaN
"	1987 Ford F150,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1854.54	4207.00	1.359	1.613	NaN	NaN
"	1987 Ford F150,PU,3 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1918.45	3550.00	1.165	1.807	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1765.24	3456.00	1.137	1.835	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1764.73	3483.00	1.138	1.834	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1762.08	3481.00	1.135	1.837	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1761.16	3475.00	1.136	1.836	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1763.91	3471.00	1.137	1.835	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1763.00	3512.00	1.134	1.838	NaN	NaN
"	1987 Ford F150,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1987	1718.04	3447.00	1.110	1.862	NaN	NaN
"	1990 Ford F150,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	1913.46	NaN	1.417	1.961	NaN	NaN
"	1990 Ford F150,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1990	1837.82	5070.00	1.407	1.978	NaN	NaN
"	1992 Ford F150 Sport,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1870.44	4023.00	1.241	1.731	NaN	NaN
"	1992 Ford F150 Sport,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1920.29	4055.00	1.254	1.718	NaN	NaN
"	1992 Ford F150 XLT,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1889.50	5324.00	1.469	1.909	NaN	NaN
"	1991 Ford F150 XLT Lariat,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1906.22	5369.00	1.480	1.898	NaN	NaN
"	1973 Ford F250,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1973	2017.33	5652.00	1.540	1.841	NaN	NaN
"	1984 Ford F250,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1899.90	NaN	1.365	2.001	NaN	NaN
"	1984 Ford F250,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1913.97	4890.00	1.407	1.971	NaN	NaN
"	1985 Ford F250,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	2611.82	7910.00	1.451	1.934	NaN	NaN
"	1991 Ford Festiva,3H,4 passengers,ballast of 556,FWD,fuel tank: F	Full-Sized	1991	1198.88	1438.00	1.091	1.208	NaN	NaN
"	1991 Ford Festiva,3H,4 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1142.71	1321.00	1.022	1.277	NaN	NaN
"	1991 Ford Festiva,3H,4 passengers,ballast of 556,FWD,fuel tank: F	Full-Sized	1991	1198.88	1453.00	1.091	1.208	NaN	NaN
"	1991 Ford Festiva,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	919.16	1128.00	0.856	1.443	NaN	NaN
"	1980 Ford LTD,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1980	1741.18	3989.00	1.236	1.670	NaN	NaN
"	1988 Ford Mustang GL,2S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	1256.07	2225.00	1.115	1.438	NaN	NaN
"	1988 Ford Mustang GT,2S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	1469.11	2620.00	1.090	1.469	NaN	NaN
"	1981 Ford Ranchero,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1981	1898.06	4579.00	1.277	1.725	NaN	NaN
"	1997 Ford Ranger,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1997	1687.97	2763.00	1.100	1.655	NaN	NaN
"	1998 Ford Ranger,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1547.91	3002.00	1.223	1.762	NaN	NaN
"	1997 Ford Ranger,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1997	1731.09	3124.00	1.130	1.625	NaN	NaN
"	1985 Ford Ranger,PU,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1985	1428.34	2306.00	1.202	1.541	NaN	NaN
"	1985 Ford Ranger,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1296.84	NaN	1.180	1.563	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1985 Ford Ranger,PU,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1985	1723.04	2906.00	1.455	1.288	NaN	NaN
"	1985 Ford Ranger,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1237.92	2119.00	1.186	1.557	NaN	NaN
"	1985 Ford Ranger,PU,3 passengers,ballast of 3114,RWD,fuel tank: 1/2	Full-Sized	1985	1733.95	NaN	1.471	1.272	NaN	NaN
"	1985 Ford Ranger,PU,3 passengers,ballast of 2224,4WD,fuel tank: 1/2	Full-Sized	1985	1831.91	NaN	1.426	1.470	NaN	NaN
"	1985 Ford Ranger,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1369.42	NaN	1.146	1.762	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1353.52	2299.00	1.141	1.602	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1501.83	2865.00	1.167	1.729	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1413.35	2705.00	1.204	1.692	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1436.90	2761.00	1.189	1.707	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1431.50	2754.00	1.188	1.708	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1431.91	2731.00	1.187	1.709	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1430.58	2739.00	1.218	1.678	NaN	NaN
"	1991 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1650.05	3440.00	1.278	1.897	NaN	NaN
"	1992 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1443.32	2490.00	1.115	1.641	NaN	NaN
"	1992 Ford Ranger ,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1566.16	3227.00	1.281	1.894	NaN	NaN
"	1985 Ford Ranger XL,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1985	1482.77	NaN	1.160	1.736	NaN	NaN
"	1992 Ford Ranger XLT,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1992	1361.67	2643.00	1.150	1.753	NaN	NaN
"	1988 Ford Taurus,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	1419.27	2687.00	0.952	1.733	NaN	NaN
"	1988 Ford Taurus,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	1489.50	2725.00	0.955	1.737	NaN	NaN
"	1992 Ford Taurus,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1449.64	2765.00	0.959	1.727	NaN	NaN
"	1987 Ford Tempo,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1201.63	2090.00	0.946	1.586	NaN	NaN
"	1987 Ford Thunderbird LX,2C,1 passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	1777.47	3335.00	1.150	1.487	NaN	NaN
"	1987 Ford Thunderbird LX,2C,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1630.58	3493.00	1.150	1.504	NaN	NaN
"	1987 Ford Thunderbird LX,2C,2 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1703.16	3238.00	1.155	1.499	NaN	NaN
"	1987 Ford Thunderbird LX,2C,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1630.07	3194.00	1.147	1.507	NaN	NaN
"	1998 Ford Windstar,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1892.05	4088.00	1.193	1.883	NaN	NaN
"	1998 Ford Windstar,VN,7 passengers,ballast of 525,FWD,fuel tank: F	Full-Sized	1998	2394.29	4929.00	1.465	1.611	NaN	NaN
"	1991 Geo Metro,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	814.78	1010.00	0.955	1.331	NaN	NaN
"	1991 Geo Tracker LSI,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1376.66	1742.00	1.161	1.036	NaN	NaN
"	1991 Geo Tracker LSI,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1080.12	1539.00	1.014	1.183	NaN	NaN
"	1991 Geo Tracker LSI,MP,4 passengers,ballast of 667,4WD,fuel tank: F	Full-Sized	1991	1447.81	1842.00	1.228	0.969	NaN	NaN
"	1991 Geo Tracker LSI,MP,4 passengers,ballast of 667,4WD,fuel tank: F	Full-Sized	1991	1450.97	1856.00	1.236	0.961	NaN	NaN
"	1991 Geo Tracker LSI,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1157.59	1560.00	1.027	1.170	NaN	NaN
"	1987 GMC 1500 Sierra,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	2042.30	NaN	1.342	2.004	NaN	NaN
"	1977 GMC 1500 Sierra Grande,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1977	1995.11	NaN	1.435	1.905	NaN	NaN
"	1985 GMC C-15 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1764.73	4407.00	1.427	1.913	NaN	NaN
"	1982 GMC C-20 Suburban,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1982	2426.81	6918.00	1.642	1.654	NaN	NaN
"	1984 GMC C-20 Suburban,MP,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	2407.75	7307.00	1.814	1.470	NaN	NaN
"	1990 GMC Jimmy ST ,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	1712.64	3122.00	1.178	1.394	NaN	NaN
"	1987 GMC Sierra,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	2621.81	NaN	1.416	1.924	NaN	NaN
"	1991 GMC Sierra C-10 1500,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1851.89	3531.00	1.242	1.743	NaN	NaN
"	1991 GMC Sierra C-10 1500,PU,3 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1990.62	3937.00	1.261	1.724	NaN	NaN
"	1991 GMC Sierra SLE 1500,PU,3 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	2121.20	4842.00	1.421	1.919	NaN	NaN
"	1991 GMC Sierra SLE 1500,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1981.55	4731.00	1.415	1.925	NaN	NaN
"	1990 GMC Suburban 1500,MP,8 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	3082.47	NaN	1.881	1.408	NaN	NaN
"	1990 GMC Suburban 1500,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	2563.30	7608.00	1.717	1.572	NaN	NaN
"	1990 GMC Suburban 1500,MP,8 passengers,ballast of 810,4WD,fuel tank: F	Full-Sized	1990	3173.60	NaN	1.947	1.342	NaN	NaN
"	1990 GMC Suburban 1500,MP,8 passengers,ballast of 810,4WD,fuel tank: F	Full-Sized	1990	3173.60	NaN	1.947	1.342	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1991 Honda Accord LX,4S,5 passengers,ballast of 200,FWD,fuel tank: F	Full-Sized	1991	1730.78	2922.00	1.258	1.460	NaN	NaN
"	1991 Honda Accord LX,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1411.52	2618.00	1.067	1.651	NaN	NaN
"	1991 Honda Accord LX,4S,5 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1710.40	2918.00	1.231	1.487	NaN	NaN
"	1991 Honda Accord LX,4S,5 passengers,ballast of 200,FWD,fuel tank: F	Full-Sized	1991	1730.78	3031.00	1.258	1.460	NaN	NaN
"	1996 Honda Acura SLX,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1996	1944.75	3902.00	1.334	1.430	NaN	NaN
"	1996 Honda Acura SLX,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1996	1996.02	3979.00	1.339	1.425	NaN	NaN
"	1996 Honda Acura SLX,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1996	2025.28	3888.00	1.325	1.439	NaN	NaN
"	1996 Honda Acura SLX,MP,4 passengers,ballast of 2477,4WD,fuel tank: F	Full-Sized	1996	2496.53	4641.00	1.547	1.217	NaN	NaN
"	1998 Honda Civic,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1143.43	1785.00	1.038	1.583	NaN	NaN
"	1981 Honda Civic,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1981	984.00	NaN	0.980	1.331	NaN	NaN
"	1983 Honda Civic,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	878.80	1216.00	0.827	1.408	NaN	NaN
"	1987 Honda Civic,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	943.12	NaN	0.948	1.427	NaN	NaN
"	1985 Honda Civic CRX,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	879.71	NaN	0.872	1.325	NaN	NaN
"	1998 Honda CR-V,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1544.55	2682.00	1.180	1.436	NaN	NaN
"	1998 Honda CR-V,MP,5 passengers,ballast of 418,4WD,fuel tank: F	Full-Sized	1998	1888.69	3055.00	1.342	1.274	NaN	NaN
"	1986 Hyundai Excel,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	938.63	1434.00	0.920	1.461	NaN	NaN
"	1987 Hyundai Excel,4S,1 passengers,ballast of Lt Ld,FWD,fuel tank: NaN	Full-Sized	1987	1262.39	2063.00	1.033	1.348	NaN	NaN
"	1987 Hyundai Excel,4S,4 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1414.27	1938.00	0.850	1.538	NaN	NaN
"	1987 Hyundai Excel,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1180.33	1778.00	0.941	1.447	NaN	NaN
"	1978 IH Scout,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1978	2062.69	3788.00	1.156	1.384	NaN	NaN
"	1991 Isuzu Amigo XL,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1613.76	2495.00	1.123	1.214	NaN	NaN
"	1986 Isuzu pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1118.65	1980.00	1.179	1.475	NaN	NaN
"	1998 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1849.34	3105.00	1.247	1.450	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1842.30	3716.00	1.359	1.410	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1848.62	3514.00	1.311	1.458	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1813.25	3638.00	1.328	1.441	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1821.00	3642.00	1.352	1.417	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1867.28	3672.00	1.339	1.430	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1865.85	3577.00	1.316	1.453	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1832.31	3494.00	1.326	1.443	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1908.97	3846.00	1.353	1.416	NaN	NaN
"	1991 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1855.05	3688.00	1.360	1.409	NaN	NaN
"	1992 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1912.64	3789.00	1.354	1.415	NaN	NaN
"	1992 Isuzu Rodeo,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1992	1905.81	3805.00	1.354	1.415	NaN	NaN
"	1994 Isuzu Trooper,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1994	2024.26	3953.00	1.351	1.410	NaN	NaN
"	1994 Isuzu Trooper,MP,4 passengers,ballast of 2545,4WD,fuel tank: F	Full-Sized	1994	2499.59	4532.00	1.543	1.218	NaN	NaN
"	1988 Isuzu Trooper,MP,2 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1813.76	3289.00	1.277	1.384	NaN	NaN
"	1988 Isuzu Trooper,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1958.41	3382.00	1.335	1.326	NaN	NaN
"	1988 Isuzu Trooper,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1660.04	3282.00	1.284	1.377	NaN	NaN
"	1984 Isuzu Trooper II,MP,5 passengers,ballast of 1446,4WD,fuel tank: 1/2	Full-Sized	1984	1952.09	NaN	1.454	1.188	NaN	NaN
"	1991 Isuzu U-15 pickup,PU,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1601.12	2684.00	1.123	1.569	NaN	NaN
"	1997 Jeep Cherokee,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1997	1662.69	2704.00	1.147	1.431	NaN	NaN
"	1997 Jeep Cherokee,MP,1 passengers,ballast of 0,4WD,fuel tank: NaN	Full-Sized	1997	NaN	NaN	NaN	NaN	NaN	NaN
"	1977 Jeep Cherokee,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1977	1991.03	3927.00	1.250	1.519	NaN	NaN
"	1984 Jeep Cherokee,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1485.42	2770.00	1.171	1.394	NaN	NaN
"	1984 Jeep Cherokee,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1565.24	2780.00	1.182	1.383	NaN	NaN
"	1984 Jeep Cherokee,MP,2 passengers,ballast of 1446,4WD,fuel tank: F	Full-Sized	1984	1783.79	2973.00	1.272	1.293	NaN	NaN
"	1984 Jeep Cherokee,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1567.99	2751.00	1.190	1.382	NaN	NaN

Reference	Description		Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1984 Jeep Cherokee,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1801.94	2923.00	NaN	NaN	NaN	NaN
"	1986 Jeep Cherokee,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1986	1467.79	2523.00	1.134	1.431	NaN	NaN
"	1987 Jeep Cherokee,MP,NaN passengers,ballast of GVWR,4WD,fuel tank: F	Full-Sized	1987	2021.00	3280.00	1.338	1.238	NaN	NaN
"	1987 Jeep Cherokee,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1523.55	2525.00	1.088	1.488	NaN	NaN
"	1987 Jeep Cherokee,MP,NaN passengers,ballast of Lt Ld,4WD,fuel tank: F	Full-Sized	1987	1702.65	2679.00	1.146	1.430	NaN	NaN
"	1988 Jeep Cherokee,MP,2 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1730.78	2851.00	1.150	1.422	NaN	NaN
"	1988 Jeep Cherokee,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1880.84	2966.00	1.231	1.341	NaN	NaN
"	1988 Jeep Cherokee,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1577.47	2812.00	1.136	1.436	NaN	NaN
"	1981 Jeep CJ-5,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1981	1196.64	1506.00	1.106	1.010	NaN	NaN
"	1981 Jeep CJ-5,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1981	1269.22	1527.00	1.134	0.982	NaN	NaN
"	1981 Jeep CJ-5,MP,4 passengers,ballast of 2224,4WD,fuel tank: 1/2	Full-Sized	1981	1703.16	NaN	1.177	0.957	NaN	NaN
"	1981 Jeep CJ-7,MP,4 passengers,ballast of 2224,4WD,fuel tank: 1/2	Full-Sized	1981	1705.40	NaN	1.304	1.071	NaN	NaN
"	1983 Jeep CJ-7,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1983	1390.72	1986.00	1.217	1.155	NaN	NaN
"	1983 Jeep CJ-7,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1983	1473.70	1978.00	1.178	1.200	NaN	NaN
"	1998 Jeep Grand Cherokee,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1804.69	3101.00	1.193	1.498	NaN	NaN
"	1998 Jeep Grand Cherokee,MP,5 passengers,ballast of 2914,4WD,fuel tank: F	Full-Sized	1998	2403.36	3986.00	1.495	1.196	NaN	NaN
"	1987 Jeep Wrangler,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1355.76	1800.00	1.067	1.295	NaN	NaN
"	1988 Jeep Wrangler,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1616.51	2092.00	1.311	1.064	NaN	NaN
"	1988 Jeep Wrangler,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1317.23	1851.00	1.168	1.207	NaN	NaN
"	1988 Jeep Wrangler,MP,2 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1469.62	1893.00	1.215	1.160	NaN	NaN
"	1990 Jeep Wrangler,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	1466.87	NaN	1.231	1.144	NaN	NaN
"	1992 Lincoln Continental,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1637.82	3402.00	1.033	1.736	NaN	NaN
"	1986 Mazda 323,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	920.49	1400.00	0.922	1.478	NaN	NaN
"	1984 Mazda B2000,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1210.19	2242.00	1.257	1.456	NaN	NaN
"	1979 Mazda GLC,3H,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1979	902.75	1390.00	1.050	1.264	NaN	NaN
"	1998 Mazda MPV,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1778.19	3200.00	1.250	1.561	NaN	NaN
"	1998 Mazda MPV,VN,7 passengers,ballast of 556,FWD,fuel tank: F	Full-Sized	1998	2283.79	3871.00	1.498	1.313	NaN	NaN
"	1991 Mazda MPV,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1926.20	3429.00	1.249	1.570	NaN	NaN
"	1998 Mazda Protégé,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1150.76	1737.00	1.005	1.596	NaN	NaN
"	1998 Mazda Protégé,4S,5 passengers,ballast of 703,FWD,fuel tank: F	Full-Sized	1998	1523.85	2182.00	1.261	1.340	NaN	NaN
"	1987 Mercedes 190,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1301.33	2095.00	1.216	1.448	NaN	NaN
"	1987 Mercedes 190,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1296.84	2113.00	1.211	1.453	NaN	NaN
"	1987 Mercedes 190 E,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1301.33	2137.00	1.211	1.442	NaN	NaN
"	1987 Mercedes 190 E,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1305.91	2142.00	1.221	1.443	NaN	NaN
"	1984 Mercury Grand Marquis,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1750.25	3907.00	1.222	1.672	NaN	NaN
"	1998 Mercury Tracer,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1223.96	1886.00	0.927	1.567	NaN	NaN
"	1998 Nissan Frontier,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1553.92	3099.00	1.291	1.663	NaN	NaN
"	1986 Nissan Maxima,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1986	1410.19	2445.00	0.884	1.666	NaN	NaN
"	1988 Nissan Maxima,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	1437.41	2462.00	0.909	1.641	NaN	NaN
"	1998 Nissan Pathfinder,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1966.36	3281.00	1.222	1.481	NaN	NaN
"	1987 Nissan Pathfinder,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1555.25	2834.00	1.205	1.444	NaN	NaN
"	1991 Nissan Pathfinder,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	1990.11	3753.00	1.273	1.375	NaN	NaN
"	1985 Nissan pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1228.34	2064.00	1.130	1.442	NaN	NaN
"	1985 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1274.21	NaN	1.146	1.430	NaN	NaN
"	1986 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1382.98	NaN	1.236	1.418	NaN	NaN
"	1988 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1988	1423.75	2446.00	1.283	1.379	NaN	NaN
"	1989 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1989	1587.05	NaN	1.125	1.517	NaN	NaN
"	1989 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1989	1410.19	2584.00	1.311	1.341	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1989 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1989	1438.74	2539.00	1.318	1.336	NaN	NaN
"	1989 Nissan pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: E	Full-Sized	1989	1431.09	2681.00	1.320	1.334	NaN	NaN
"	1998 Nissan Sentra,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1203.67	1848.00	0.959	1.573	NaN	NaN
"	1983 Nissan Sentra,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	965.85	NaN	0.972	1.435	NaN	NaN
"	1987 Nissan Sentra,2S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	970.34	1461.00	0.931	1.500	NaN	NaN
"	1987 Nissan Sentra,2S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	956.78	1461.00	0.922	1.509	NaN	NaN
"	1985 Nissan Stanza,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1206.12	NaN	0.968	1.509	NaN	NaN
"	1987 Nissan Van,VN,0 passengers,ballast of 0,RWD,fuel tank: F?	Full-Sized	1987	1528.13	2418.00	0.965	1.385	NaN	NaN
"	1987 Nissan XE King Cab,PU,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1987	1980.12	3659.00	1.614	1.335	NaN	NaN
"	1987 Nissan XE King Cab,PU,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	1579.82	3066.00	1.363	1.586	NaN	NaN
"	1987 Nissan XE King Cab,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1387.56	2808.00	1.340	1.609	NaN	NaN
"	1980 Oldsmobile 98,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1980	1888.07	4984.00	1.312	1.711	NaN	NaN
"	1976 Oldsmobile 98 Regency,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1976	2298.88	6399.00	1.554	1.662	NaN	NaN
"	1990 Oldsmobile Cutlass Calais,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1278.29	2082.00	0.903	1.739	NaN	NaN
"	1990 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1403.36	2285.00	0.945	1.697	NaN	NaN
"	1990 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1333.13	2142.00	0.907	1.735	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1240.57	2050.00	0.949	1.693	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1358.00	2200.00	0.909	1.733	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1304.99	2138.00	0.931	1.711	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1305.40	2133.00	0.923	1.719	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1305.91	2117.00	0.924	1.718	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1302.75	2125.00	0.923	1.719	NaN	NaN
"	1991 Oldsmobile Cutlass Calais,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1304.99	2136.00	0.926	1.716	NaN	NaN
"	1985 Oldsmobile Cutlass Ciera,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1278.70	2407.00	0.973	1.691	NaN	NaN
"	1985 Oldsmobile Cutlass Ciera,4S,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1985	1316.82	2531.00	0.969	1.698	NaN	NaN
"	1985 Oldsmobile Cutlass Ciera,4S,4 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1577.06	2794.00	1.118	1.549	NaN	NaN
"	1985 Oldsmobile Cutlass Ciera,4S,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1985	1316.82	2547.00	0.969	1.698	NaN	NaN
"	1985 Oldsmobile Cutlass Ciera,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1361.67	2629.00	1.007	1.660	NaN	NaN
"	1980 Plymouth Arrow,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1980	1240.16	2504.00	1.244	1.545	NaN	NaN
"	1998 Plymouth Grand Voyager,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1898.57	4356.00	1.288	1.752	NaN	NaN
"	1985 Plymouth Reliant,SW,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1203.87	2161.00	0.975	1.578	NaN	NaN
"	1987 Plymouth Sundance,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1169.83	1866.00	0.945	1.519	NaN	NaN
"	1991 Plymouth Voyager,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1726.20	3460.00	1.165	1.680	NaN	NaN
"	1991 Plymouth Voyager,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	1813.25	4177.00	1.256	1.792	NaN	NaN
"	1992 Plymouth Voyager,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1992	1674.11	3424.00	1.150	1.695	NaN	NaN
"	1990 Plymouth Voyager SE,VN,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1990	1732.11	NaN	1.240	1.795	NaN	NaN
"	1984 Pontiac Fiero,2C,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1984	1256.07	NaN	1.312	1.063	NaN	NaN
"	1985 Pontiac Fiero,2C,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1287.77	NaN	1.347	1.015	NaN	NaN
"	1985 Pontiac Fiero,2C,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1256.07	1619.00	1.389	0.986	NaN	NaN
"	1985 Pontiac Fiero,2C,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1246.99	NaN	1.346	1.042	NaN	NaN
"	1985 Pontiac Grand Am,2C,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1985	1165.34	1999.00	0.881	1.753	NaN	NaN
"	1989 Pontiac Grand Am,2C,1 passengers,ballast of 0,FWD,fuel tank: E	Full-Sized	1989	1285.93	2247.00	1.060	1.569	NaN	NaN
"	1978 Pontiac LeMans,2C,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1978	1496.33	3152.00	1.247	1.496	NaN	NaN
"	1988 Pontiac LeMans,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	938.63	1412.00	0.998	1.522	NaN	NaN
"	1982 Renault LeCar,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1982	915.90	NaN	1.014	1.424	NaN	NaN
"	1998 Saturn SL,4S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1126.91	1786.00	1.044	1.554	NaN	NaN
"	1984 Subaru Brat,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1984	1043.32	1688.00	0.988	1.455	NaN	NaN
"	1991 Subaru Justy GL,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1991	958.10	1246.00	0.877	1.396	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1987 Subaru XT Coupe,2C,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1035.17	1677.00	0.943	1.508	NaN	NaN
"	1988 Suzuki Samurai,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1228.85	1341.00	1.162	0.870	NaN	NaN
"	1988 Suzuki Samurai,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	933.64	1138.00	1.007	1.025	NaN	NaN
"	1988 Suzuki Samurai,MP,4 passengers,ballast of 1112,4WD,fuel tank: 1/2	Full-Sized	1988	1331.70	NaN	1.157	0.875	NaN	NaN
"	1988 Suzuki Samurai,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	936.39	1144.00	1.138	0.894	NaN	NaN
"	1988 Suzuki Samurai,MP,2 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1090.93	1197.00	1.028	1.004	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1007.54	1210.00	1.016	1.016	NaN	NaN
"	1988 Suzuki Samurai,MP,2 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1078.70	1138.00	1.039	0.993	NaN	NaN
"	1988 Suzuki Samurai,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	940.88	1160.00	1.005	1.027	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1007.54	1208.00	1.018	1.014	NaN	NaN
"	1988 Suzuki Samurai,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1241.49	1360.00	1.167	0.865	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1009.38	1230.00	1.020	1.018	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1009.38	1203.00	1.020	1.018	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: E	Full-Sized	1988	978.49	1143.00	0.984	1.061	NaN	NaN
"	1988 Suzuki Samurai,MP,4 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1225.18	1380.00	1.172	0.873	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1010.30	1182.00	1.027	1.018	NaN	NaN
"	1988 Suzuki Samurai,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1988	1007.54	1192.00	1.024	1.021	NaN	NaN
"	1990 Toyota 4Runner,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	1999.59	3749.00	1.126	1.495	NaN	NaN
"	1990 Toyota 4Runner,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1990	NaN	NaN	NaN	NaN	NaN	NaN
"	1998 Toyota 4Runner,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1998	1905.71	3246.00	1.226	1.444	NaN	NaN
"	1998 Toyota 4Runner,MP,5 passengers,ballast of 1703,4WD,fuel tank: F	Full-Sized	1998	2380.63	3842.00	1.422	1.248	NaN	NaN
"	1987 Toyota 4Runner,MP,NaN passengers,ballast of GVWR+,4WD,fuel tank: F	Full-Sized	1987	2348.83	3578.00	1.479	1.145	NaN	NaN
"	1987 Toyota 4Runner,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1987	1592.05	3331.00	1.226	1.398	NaN	NaN
"	1987 Toyota 4Runner,MP,NaN passengers,ballast of Lt Ld,4WD,fuel tank: F	Full-Sized	1987	1791.03	2972.00	1.236	1.388	NaN	NaN
"	1989 Toyota 4Runner,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1989	1675.94	3042.00	1.277	1.352	NaN	NaN
"	1989 Toyota 4Runner,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1989	1666.87	NaN	1.283	1.359	NaN	NaN
"	1983 Toyota Camry,5H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1116.41	2036.00	1.034	1.567	NaN	NaN
"	1983 Toyota Camry,5H,4 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1402.45	2227.00	1.183	1.433	NaN	NaN
"	1983 Toyota Camry,5H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1983	1183.89	1874.00	1.068	1.548	NaN	NaN
"	1987 Toyota Camry,4S,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	1319.06	2404.00	1.016	1.581	NaN	NaN
"	1976 Toyota Corolla,2C,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1976	1066.46	1706.00	1.054	1.318	NaN	NaN
"	1987 Toyota Corolla FX,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	995.72	1594.00	0.944	1.487	NaN	NaN
"	1985 Toyota Coventry,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1985	1505.40	NaN	0.916	1.319	NaN	NaN
"	1982 Toyota Cressida,4S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1982	1310.40	2361.00	1.194	1.450	NaN	NaN
"	1979 Toyota Land Cruiser,MP,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1979	1960.24	3930.00	1.364	1.328	NaN	NaN
"	1991 Toyota Land Cruiser,MP,1 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1991	2318.86	4505.00	1.377	1.481	NaN	NaN
"	1987 Toyota LE Van,VN,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1510.40	2193.00	0.931	1.311	NaN	NaN
"	1987 Toyota LE Van,VN,NaN passengers,ballast of Lt Ld,RWD,fuel tank: F	Full-Sized	1987	1701.73	2374.00	0.817	1.425	NaN	NaN
"	1987 Toyota LE Van,VN,NaN passengers,ballast of GVWR,RWD,fuel tank: F	Full-Sized	1987	2155.15	3218.00	1.115	1.127	NaN	NaN
"	1986 Toyota MR2,2C,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1071.05	1457.00	1.314	1.005	NaN	NaN
"	1986 Toyota MR2,2C,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1089.60	1421.00	1.284	1.035	NaN	NaN
"	1989 Toyota pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1989	1255.15	2560.00	1.216	1.642	NaN	NaN
"	1991 Toyota Previa LE,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1782.47	3135.00	1.329	1.529	NaN	NaN
"	1986 Toyota RN50 pickup,PU,2 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1283.69	2118.00	1.145	1.478	NaN	NaN
"	1986 Toyota RN50 pickup,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1986	1225.18	1962.00	1.151	1.472	NaN	NaN
"	1988 Toyota RN50 pickup,PU,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1988	1202.96	2138.00	1.158	1.463	NaN	NaN
"	1986 Toyota RN60 pickup,PU,0 passengers,ballast of 0,4WD,fuel tank: F	Full-Sized	1986	1433.74	2383.00	1.095	1.526	NaN	NaN
"	1983 Toyota Starlet,3H,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1983	915.90	NaN	1.030	1.281	NaN	NaN

Reference	Description	Vehicle Type	Mass [kg]	Moment of Inertia [kg.m2]	Distance, C.G. to F. Axle [m]	Distance, C.G. to R. Axle [m]	Front cornering Stiffness [m.kg.s-2]	Rear cornering Stiffness [m.kg.s-2]	Published Test Speed, [m.s-1]
"	1998 Toyota Tacoma,PU,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1998	1436.29	3024.00	1.356	1.745	NaN	NaN
"	1998 Toyota Tacoma,PU,5 passengers,ballast of 2972,RWD,fuel tank: F	Full-Sized	1998	2040.57	4035.00	1.652	1.449	NaN	NaN
"	1998 Toyota Tercel,2S,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1998	1053.21	1473.00	0.953	1.422	NaN	NaN
"	1998 Toyota Tercel,2S,5 passengers,ballast of 58,FWD,fuel tank: F	Full-Sized	1998	1360.35	1689.00	1.138	1.237	NaN	NaN
"	1971 Volkswagen Beetle,2S,0 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1971	856.57	1289.00	1.412	0.996	NaN	NaN
"	1987 Volkswagen Vanagon,VN,7 passengers,ballast of 2780,RWD,fuel tank: 1/2	Full-Sized	1987	2388.28	NaN	1.280	1.184	NaN	NaN
"	1987 Volkswagen Vanagon GL,VN,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1987	1682.26	NaN	1.146	1.318	NaN	NaN
"	1991 Volvo 240,4S,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1468.20	2663.00	1.303	1.344	NaN	NaN
"	1991 Volvo 740,4S,1 passengers,ballast of 0,RWD,fuel tank: F	Full-Sized	1991	1500.00	2845.00	1.315	1.464	NaN	NaN
"	1987 Yugo GV,3H,0 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1987	820.69	940.00	0.743	1.391	NaN	NaN
"	1988 Yugo GV,3H,1 passengers,ballast of 0,FWD,fuel tank: F	Full-Sized	1988	924.06	1073.00	0.787	1.372	NaN	NaN

C.4 References

1. E. Ahring and M. Mitschke, “Comparison of All-Wheel Steerings in the System Driver-Vehicle,” *Vehicle System Dynamics*, vol. 24, pp. 283-298, 1995.
2. A. Alleyne, “A Comparison of Alternative Obstacle Avoidance Strategies for Vehicle Control,” *Vehicle System Dynamics*, vol. 27, pp. 371-392, 1997.
3. R. T. Bundorf, “A Primer on Vehicle Directional Control,” General Motors Technical Center, Warren, Michigan Engineering Publication A-2739, September 19 1968.
4. Y. H. Cho and J. Kim, “Design of Optimal Four-Wheel Steering System,” *Vehicle System Dynamics*, vol. 24, pp. 661-682, 1995.
5. C. Doniselli, G. Mastinu, and M. Gobbi, “Aerodynamic Effects on Ride Comfort and Road Holding of Automobiles,” *Vehicle System Dynamics Supplement*, vol. 25, pp. 1996, 1996.
6. H. e. a. Harada, “Control Effects of Active Rear-Wheel-Steering on Driver Vehicle Systems,” presented at AVEC'96, Aachen, 1996.

7. C. Hatipoglu, K. Redmill, and U. Ozguner, "Automated Lane Change: Theory and Practice," presented at Advances in Automotive Control 1998, Proceedings of the 2nd IFAC Workshop, Mohican State Park, Loudonville, Ohio, USA, 1998.
8. S. Horiuchi, N. Yuhara, and A. Takei, "Two Degree of Freedom H-infinity Controller Synthesis for Active Four Wheel Steering Vehicles," *Vehicle System Dynamics Supplement*, vol. 25, pp. 275-292, 1996.
9. D. J. LeBlanc, P. Benhovens, C. F. Lin, T. Pilutti, R. Ervin, A. G. Ulsoy, C. MacAdam, and G. Johnson, "A Warning and Intervention System for Preventing Road Departure Accidents," in *The Dynamics of Vehicles on Roads and Tracks, Proceedings of the 14th IAVSD Symposium*, vol. 25. Ann Arbor, MI, 1995, pp. 383-396.
10. Y. Lin, "Improving Vehicle Handling Performance by a Closed-Loop 4WS Driving Controller," *SAE Paper #921604*, pp. 1447-1457, 1992.
11. N. Matsumoto and M. Tomizuka, "Vehicle Lateral Velocity and Yaw Rate Control with Two Independent Control Inputs," *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 114, pp. 606-613, 1992.
12. M. Nagai, E. Ueda, and A. Moran, "Nonlinear Design Approach to Four-Wheel-Steering Systems Using Neural Networks," *Vehicle System Dynamics*, pp. 329-342, 1995.
13. M. Nagai, Y. Hirano, and S. Yamanaka, "Integrated Control of Active Rear Steering and Direct Yaw Moment Control," *Vehicle System Dynamics*, vol. 27, pp. 357-370, 1997.
14. M. Nagai, Y. Hirano, and S. Yamanaka, "Integrated Robust Control of Active Rear Wheel Steering and Direct Yaw Moment Control," *Vehicle System Dynamics Suppliment*, vol. 28, pp. 416-421, 1998.
15. L. Palkovics, "Effect of the Controller Parameters on the Steerability of the Four Wheel Steered Car," *Vehicle System Dynamics*, vol. 21, pp. 109-128, 1992.
16. H. Peng and M. Tomizuka, "Preview Control for Vehicle Lateral Guidance in Highway Automation," *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 115, pp. 679-686, 1993.
17. H. Peng, W. Zhang, M. Tomizuka, and S. Shladover, "A Reusability Study of Vehicle Lateral Control system," *Vehicle System Dynamics*, vol. 28, pp. 259-278, 1994.

18. U. N. Petersen, A. Rukgauer, and W. O. Schiehlen, “Lateral Control of a Convoy Vehicle System,” *Vehicle System Dynamics*, vol. 25, pp. 519-532, 1996.
19. T. Pilutti, G. Ulsoy, and D. Hrovat, “Vehicle Steering Intervention Through Differential Braking,” presented at Proceedings of the American Control Conference, Seattle, WA, 1995.
20. Y. Shibahata, K. Shimada, and T. Tomari, “Improvement of Vehicle Maneuverability by Direct Yaw Moment Control,” *Vehicle System Dynamics*, vol. 22, pp. 465-481, 1993.
21. Z. Shiller and S. Sundar, “Emergency Lane-Change Maneuvers of Autonomous Vehicles,” *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 120, pp. 37-44, 1998.
22. D. E. Smith and R. E. Benton, “Automated Emergency Four-Wheel-Steered Vehicle Using Continuous Gain Equations,” *Vehicle System Dynamics*, vol. 26, pp. 127-142, 1996.
23. D. E. Smith and J. M. Starkey, “Effects of Model Complexity on the Performance of Automated Vehicle Steering Controllers: Model Development, Validation and Comparison,” *Vehicle System Dynamics*, vol. 24, pp. 163-181, 1995.
24. J. Sridhar and H. Hatwal, “A Comparative Study of Four Wheel Steering Models Using the Inverse Solution,” *Vehicle System Dynamics*, vol. 22, pp. 1-17, 1992.
25. Y. Tagawa, H. Ogata, K. Morita, M. Nagai, and H. Mori, “Robust Active Steering System Taking Account of Nonlinear Dynamics,” *Vehicle System Dynamics Suppliment*, vol. 25, pp. 668-681, 1996.
26. K. A. Unyelioglu, C. Hatipoglu, and U. Ozguner, “Design and Stability Analysis of a Lane Following Controller,” *IEEE Transactionis on Control Systems Technology*, vol. 5, pp. 127-134, 1997.
27. A. B. Will and S. H. Zak, “Modeling and Control of an Automated Vehicle,” *Vehicle System Dynamics*, vol. 27, pp. 131-155, 1997.
28. B. A. Guvenc, T. Bunte, D. Odenthal, and L. Guvenc, “Robust Two Degree of Freedom Vehicle Steering Controller Design,” presented at 2001 American Control Conferenc e, Washington, D.C., 2001.

29. D. M. Bevly, R. Sheridan, and J. C. Gerdes, “Integrating INS Sensors with GPS Velocity Measurements for Continuous Estimation of Vehicle Sideslip and Tire Cornering Stiffness,” presented at 2001 American Control Conference, Washington, D.C., 2001.
30. A. Shrivastava and R. Rajamani, “Fault Diagnostics for GPS-Based Lateral Vehicle Control,” presented at 2001 American Control Conference, Washington, D.C., 2001.
31. S. Mammar, D. Koenig, and L. Nouveliere, “Combination of Feedforward and Feedback H-infinity Control For Speed Scheduled Vehicle Automatic Steering,” presented at 2001 American Control Conference, Washington, D.C., 2001.
32. J. Huang and M. Tomizuka, “H-infinity Controller for Vehicle Lateral Control Under Fault in Front or Rear Sensors,” presented at 2001 American Control Conference, Washington, D.C., 2001.
33. J. R. Zhang, A. Rachid, and S. J. Xu, “Velocity Controller Design for Automatic Steering of Vehicles,” presented at 2001 American Control Conference, Washington, D.C., 2001.
34. B. Samadi, R. Kazemi, K. Y. Nikravesh, and M. Kabganian, “Real-Time Estimation of Vehicle State and Tire-Road Friction Forces,” presented at 2001 American Control Conference, Washingtion, D.C., 2001.
35. S. Mammar and B. B. Baghdassarian, “Two-degree-of-freedom Formulation of Vehicle Handling Improvement by Active Steering,” presented at 2000 American Control Conference, Chicago, IL, 2000.
36. S. E. Lyshevski and A. Nazarov, “Lateral Maneuvering of Ground Vehicles: Modeling and Control,” presented at 2000 American Control Conference, Chicago, Illinois, 2000.
37. H.-S. Tan and C.-Y. Chan, “Design of Steering Controller and Analysis of Vehicle Lateral Dynamics under Impulsive Disturbances,” presented at 2000 American Control Conference, Chicago, IL, 2000.
38. K.-T. Feng, H.-S. Tan, M. Tomizuka, and W.-B. Zhang, “Look-ahead Human-machine Interface for Assistance of Manual Vehicle Steering,” presented at 1999 American Controls Conference, San Diego, CA, 1999.
39. C. Chen and H.-S. Tan, “Experimental Study of Dynamic Look-Ahead Scheme for Vehicle Steering Control,” presented at 1999 American Control Conference, San Diego, CA, 1999.

40. C.-Y. Chan and H.-S. Tan, "Lane Tracking Control in Vehicle-Following Collision Situations," presented at 1999 American Control Conference, San Diego, CA, 1999.

41. K.-T. Feng, H.-S. Tan, and M. Tomizuka, "Automatic Steering Control of Vehicle Lateral Motion with the Effect of Roll Dynamics," presented at 1998 American Control Conference, Philadelphia, PA, 1998.

42. P. Hingwe and M. Tomizuka, "Experimental Evaluation of a Chatter Free Sliding Mode Control for Lateral Control in AHS," presented at 1997 American Control Conference, Albuquerque, NM, 1997.

43. N. A. El-Esnawy and J. F. Wilson, "Lateral Dynamics and Stability of Two Full Vehicles in Tandem," *Transactions of the ASME*, vol. 120, pp. 50-56, 1998.

44. A. Y. Lee, "Design of Stability Augmentation Systems for Automotive Vehicles," *Journal of Dynamic Systems, Measurement, and Control*, vol. 112, pp. 489-495, 1990.

45. N. Sivashankar and A. G. Ulsoy, "Yaw Rate Estimation for Vehicle Control Applications," *Journal of Dynamic Systems, Measurement, and Control*, vol. 120, pp. 267-274, 1998.

46. A. Modjtahedzadeh and R. A. Hess, "A Model of Driver Steering Control Behavior for Use in Assessing Vehicle Handling Qualities," *Transactions of the ASME*, vol. 115, pp. 456-464, 1993.

47. A. Schiepatti, "Notes on Vehicle Stability Parameters with Respect to Active Safety," *In Granger, Unknown, have no copy*, vol. 2, pp. 119-126.

48. T. Shiotsuka, A. Nagamatsu, and K. Yoshida, "Adaptive Control of 4WS System by Using Neural Network," *Vehicle System Dynamics*, pp. 411-424, 1993.

49. D. E. Smith and J. M. Starkey, "Effects of Model Complexity on the Performance of Automated Vehicle Steering Controllers: Controller Development and Evaluation," *Vehicle System Dynamics*, pp. 627-645, 1994.

50. A. Alleyne, "A Comparison of Alternative Intervention Stragies for Unintended Roadway Departure (URD) Control," *Vehicle System Dynamics*, vol. 27, pp. 1570186, 1997.

51. Y. H. Cho and J. Kim, "Stability Analysis of the Human Controlled Vehicle Moving Along a Curved Path," *Vehicle System Dynamics*, vol. 25, pp. 51-69, 1996.

52. K.-T. Feng, H.-S. Tan, and M. Tomizuka, "Design of Vehicle Lateral Guidance System for Driver Assistance," presented at American Control Conference, Chicago, IL, 2000.

53. C. Gerdes, “Personal Communication regarding 1999 Mercedes E320 Sedan Bicycle Model Parameters,” , 2002.

54. B. Jang and D. Karnopp, “Simulation of Vehicle and Power Steering Dynamics Using Tire Model Parameters Matched to Whole Vehicle Experimental Results,” *Vehicle System Dynamics*, vol. 33, pp. 121-133, 2000.

55. D. J. LeBlanc, G. E. Johnson, P. J. T. Venhovens, G. Gerber, R. DeSonia, R. D. Ervin, C.-F. Lin, A. G. Ulsoy, and T. E. Pilutti, “CAPC: A Road-Departure Prevention System,” *IEEE Control Systems Magazine*, vol. December, pp. 61-71, 1996.

56. Y. Lee and S. H. Zak, “Genetic Fuzzy Tracking Controllers for Autonomous Ground Vehicles,” presented at American Control Conference, Anchorage, AK, 2002.

57. G. Lu and M. Tomizuka, “Vehicle Lateral Control with Combined Use of a Laser Scanning Radar Sensor and Rear Magnetometers,” presented at American Control Conference, Anchorage, AK, 2002.

58. A. Alleyne and M. DePoorter, “Lateral Displacement Sensor Placement and Forward Velocity Effects on Stability of Lateral Control of Vehicles,” , Albuquerque, NM, 1997.

59. W. Langson and A. Alleyne, “Multivariable Bilinear Vehicle Control using Steering and Individual Wheel Torques,” presented at American Control Conference, Albuquerque, NM, 1997.

60. T. Pilutti, G. Ulsoy, and D. Hrovat, “Vehicle Steering Intervention Through Differential Braking,” *Transactions of the ASME*, vol. 120, pp. 314-321, 1998.

61. R. S. Sharp and V. Valtetsiotis, “Optimal Preview Car Steering Control,” *Vehicle System Dynamics Supplement*, vol. 35, pp. 101-117, 2001.

62. M. Russo, R. Russo, and A. Volpe, “Car Parameters Identification by Handling Manoeuvres,” *Vehicle System Dynamics*, vol. 34, pp. 423-436, 2000.

63. R. S. J. Rice and W. F. J. Milliken, .

64. H.-S. Tan, B. Bougler, and W.-B. Zhang, “Automatic Steering Based on Roadway Markers: From Highway Driving to Precision Docking,” *Vehicle System Dynamics*, vol. 37, pp. 315-338, 2002.

65. P. J. T. Venhovens and K. Naab, “Vehicle Dynamics Estimation Using Kalman Filters,” *Vehicle System Dynamics*, vol. 32, pp. 171-184, 1999.

66. G. J. Heydinger, R. A. Bixel, W. R. Garrot, M. Pyne, J. G. Howe, and D. A. Guenther, "Measured Vehicle Inertial Parameters - NHTSA's Data Through November 1998," *Society of Automotive Engineers*, vol. SAE Paper No. 930897, 1999.

Vita

Sean Brennan was born on Valentine's Day, 1974, in El Paso, Texas and grew up in Las Cruces, New Mexico. Sean attended New Mexico State University from 1994-1997, earning two Bachelor degrees: one in Physics and one in Mechanical Engineering. Graduating at the top of both departments, he was honored as the outstanding graduate in the College of Engineering. While at NMSU, Sean was a TA for a Fluids and Heat-Transfer class and worked in an electrical contacts laboratory developing automation systems to test the stability of electrical connections undergoing mechanical motion for nuclear missiles and consumer automobiles. His senior project on the modification of snow-plow trucks to 'harvest' tumbleweeds from highways won numerous design awards and was featured on CNN, the Washington Post, and in ASEE magazine.

Sean entered the University of Illinois at Urbana-Champaign in June 1997 working under the guidance of Professor Andrew G. Alleyne, and was awarded the National Science Foundation Graduate Fellowship the following spring. Sean served as a TA in the Mechatronics class in 1999 and in the Industrial Control class in 2001. He earned his Master's Degree in 1999 on the topic of "Modeling and Control Issues Associated with Scaled Vehicles." To date, Sean has published 12 conference and journal articles on his work and has won numerous presentation awards. Notably, one of his papers was featured as the cover article of Control Systems Magazine in June 2001.

While his wife is finishing her Ph.D., Sean is currently working as a consultant to the College of Engineering, assisting in developing new control systems for the undergraduate laboratories. After graduation, Sean and his wife will be joining Pennsylvania State University in June 2003 as tenure-track Assistant Professors.